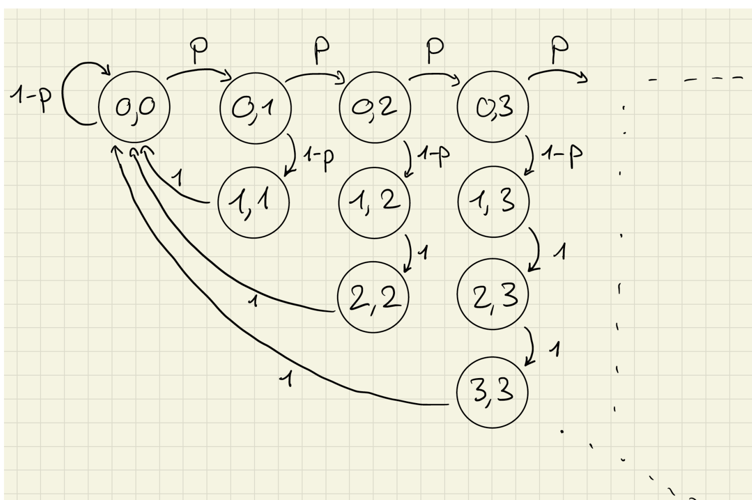
Midterm Solutions

Exercise 1. (10 points)

Consider the Markov chain $(X_n,\, n \geq 0)$ with state space $S = \{(j,k),\, j,k \in \mathbb{N},\, j \leq k\}$ and transition probabilities given by

$$\begin{cases} p_{(0,0),(0,0)} = 1-p & p_{(0,k),(0,k+1)} = p \quad \text{for } k \geq 0 \\ p_{(0,k),(1,k)} = 1-p \quad \text{for } k \geq 1 & p_{(j,k),(j+1,k)} = 1 \quad \text{for } 1 \leq j \leq k \\ p_{(k,k),(0,0)} = 1 \quad \text{for } k \geq 1 & \end{cases}$$

where $0 < p < 1$. Here is the corresponding transition graph for the first states:



a) (1 point) Compute $f_{(0,0),(0,0)}(n) = \mathbb{P}(T_{(0,0)} = n \,|\, X_0 = (0,0))$ for a generic value of $n \geq 1$.

NB: $T_{(0,0)} = \inf\{n \geq 1 : X_n = (0,0)\}$.

Answer: First note that the chain is irreducible, and that starting in $(0,0)$, it is only possible to come back to $(0,0)$ in an odd number of steps, and for each odd value of $n \geq 1$, there is only a single path. So

$$f_{(0,0),(0,0)}(n) = 0 \text{ if } n \text{ is even} \quad \text{and} \quad f_{(0,0),(0,0)}(n) = p^{(n-1)/2}\,(1-p) \text{ if } n \text{ is odd}$$

b) (2 points) Prove that the chain X is recurrent.

Answer: The chain is recurrent, as

$$f_{(0,0),(0,0)} = \sum_{n\geq 1} f_{(0,0),(0,0)}(n) = \sum_{k\geq 0} f_{(0,0),(0,0)}(2k+1) = \sum_{k\geq 0} p^k\,(1-p) = \frac{1-p}{1-p} = 1$$

c) (2 points) Prove that the chain X is also positive-recurrent.

Hint: Compute $\mathbb{E}(T_{(0,0)} \,|\, X_0 = (0,0))$, using the identity: $\sum_{k\geq 0} k\, p^k = \frac{p}{(1-p)^2}$ valid for $0 < p < 1$.

Answer: The computation gives

$$\mathbb{E}(T_{(0,0)} \,|\, X_0 = (0,0)) = \sum_{n\geq 1} n\, \mathbb{P}(T_{(0,0)} = n \,|\, X_0 = (0,0)) = \sum_{k\geq 0} (2k+1)\, f_{(0,0),(0,0)}(2k+1)$$

$$= \sum_{k\geq 0} (2k+1)\, p^k\, (1-p) = \left(2\, \frac{p}{(1-p)^2} + \frac{1}{1-p}\right)(1-p) = \frac{2p+1-p}{1-p} = \frac{1+p}{1-p} < +\infty$$

so the chain is positive-recurrent.

d) (3 points) Compute the stationary distribution π of the chain X.

Answer: By the theorem seen in class, $\pi_{(0,0)} = \frac{1}{\mathbb{E}(T_{(0,0)} \,|\, X_0 = (0,0))} = \frac{1-p}{1+p}$. Also, we have

$$\pi_{(0,k+1)} = \pi_{(0,k)}\, p \quad \text{so by induction, we obtain} \quad \pi_{(0,k)} = p^k\, \pi_{(0,0)} \quad \text{for } k \geq 1$$

By a similar reasoning, we obtain for all $1 \leq j \leq k$:

$$\pi_{(j,k)} = (1-p)\, \pi_{(0,k)} = (1-p)\, p^k\, \pi_{(0,0)}$$

and one checks that indeed

$$\sum_{k\geq 0} \pi_{(0,k)} + \sum_{k\geq j\geq 1} \pi_{(j,k)} = 1$$

e) (1 point) Is π also a limiting distribution? Justify.

Answer: Yes, as the chain is not only irreducible and positive-recurrently, but also aperiodic (self-loop in $(0,0)$).

f) (1 point) Does detailed balance hold? Justify.

Answer: No, as there are one-way arrows in the transition graph.

Exercise 2. (10 points)

Let $N \geq 5$ be an integer and consider two Markov chains $(X_n,\, n \geq 0)$, $(Y_n,\, n \geq 0)$ defined on the same state space $S = \{0, 1, 2, \ldots, N-1\}$ and with the same transition matrix $P = (p_{ij})_{i,j\in S}$, where

$$p_{ij} = \begin{cases} p & \text{if } j = i+1 \,(\text{mod } N) \\ q & \text{if } j = i-1 \,(\text{mod } N) \\ 0 & \text{otherwise} \end{cases}$$

and $p, q > 0$ are such that $p + q = 1$. Let also $(Z_n,\, n \geq 0)$ be the Markov chain on S defined as follows:

$$Z_n = Y_n - X_n \,(\text{mod } N) \quad \text{for } n \geq 0$$

a) (2 points) Compute the transition matrix $Q = (q_{ij})_{i,j \in S}$ of the chain Z.

Answer: The computation gives

$$q_{ij} = \begin{cases} p^2 + q^2 & \text{if } j = i \\ pq & \text{if } j - i = +2 \,(\text{mod } N) \\ pq & \text{if } j - i = -2 \,(\text{mod } N) \end{cases}$$

which is a symmetric matrix, even if P isn't.

b) (2 points) Compute the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{N-1}$ of the matrix Q (for a generic value of N).

Hint: If $A = \text{circ}(c_0, c_1, \ldots, c_{N-1})$ is an $N \times N$ circulant matrix, i.e.

$$A = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{N-2} & c_{N-1} \\ c_{N-1} & c_0 & c_1 & & & c_{N-2} \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ c_2 & & & c_{N-1} & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{N-2} & c_{N-1} & c_0 \end{pmatrix}$$

then its eigenvalues are given by

$$\lambda_k = \sum_{j=0}^{N-1} c_j \, \exp(2\pi i j k/N) \quad k = 0, \ldots, N-1$$

Please note that with this notation, the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{N-1}$ are not ordered.

Answer: The computation gives

$$\lambda_k = p^2 + q^2 + 2pq\cos(4\pi k/N) \quad k = 0, \ldots, N-1$$

Note that all these (unordered) eigenvalues are non-negative (irrespective of the values of p and q).

c) (3 points) For what values of $N \geq 5$ is the Markov chain Z ergodic? For these values, compute the limiting and stationary distribution of the chain. Is detailed balance satisfied? *Justify all your answers.*

Answer: In order for the chain to be irreducible, N needs to be odd, in which case the chain is also aperiodic, and therefore ergodic (as it is also finite). Because the matrix Q is doubly-stochastic, the stationary distribution is uniform, and detailed balance holds, as Q is also symmetric.

d) (2 points) For the values of N found in part c), compute the spectral gap γ of the chain Z.

Answer: The eigenvalue which is the closest to $+1$ is the one with $k = (N \pm 1)/2$, so

$$\gamma = 1 - (p^2 + q^2 + 2pq \, \cos(2\pi/N) = 1 - p^2 - (1-p)^2 - 2p(1-p) \, \cos(2\pi/N) = 2p(1-p)\,(1 - \cos(2\pi/N))$$

e) (1 point) For large values of N (still satisfying the condition found in part c), how large should n be (approximately) in order to ensure that the distribution of Z_n is ε-close (in total variation distance) to the stationary distribution?

Hint: You may use the approximation $\cos(x) \simeq 1 - \frac{x^2}{2}$, valid for small values of x.

Answer: The spectral gap is given in this case by $\gamma \simeq 2p(1-p)\, 2\pi^2/N^2$, so in order to ensure that the total variation distance is close to zero, at least $\Theta(N^2)$ steps (and more precisely $\Theta(N^2 \log N)$ steps) are needed.

BONUS f) (2 points) What is the average time between two encounters of the chains X and Y? Does it depend on the values of N, p and q? If yes, how?

Answer: When N is odd, this average time between two encounters of X and Y is N, because the stationary distribution of the chain Z is uniform on $S = \{0, 1, \ldots, N-1\}$.

When N is even, the chain Z is not ergodic and only visits states with the same parity, so in this case, the average time between two encounters of X and Y is $N/2$.

So these average times do depend on the value of N, but not on the values of p and q.