Astrophysics III: Stellar and galactic dynamics

Solutions

Problem 1:

We are dealing with the Hamiltonian of the form

$$H(q,p) = \frac{1}{2}\vec{p}^2 + \Phi(\vec{q}) - \vec{\Omega}\cdot(\vec{q}\times\vec{p}) \tag{1}$$

We set the rotation to be along the z axis, and for it to be uniformly rotating, it needs to be constant, i.e.

$$\vec{\Omega} = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix} \qquad \Rightarrow \vec{\Omega}\cdot(\vec{q}\times\vec{p}) = \Omega(q_x p_y - p_x q_y)$$

The equations of motion in canonical coordinates are given by Hamilton's equations:

$$\dot{p} = -\frac{\partial}{\partial q}H(p,q), \quad \dot{q} = \frac{\partial}{\partial p}H(p,q) \tag{2}$$

in our case:

$$\begin{aligned}
\dot{q}_x &= p_x + \Omega q_y \\
\dot{q}_y &= p_y - \Omega q_x \\
\dot{p}_x &= -\frac{\partial}{\partial q_x}\Phi(q,p) + \Omega p_y \\
\dot{p}_y &= -\frac{\partial}{\partial q_y}\Phi(q,p) - \Omega p_x
\end{aligned}$$

The relations between cartesian and canonical coordinates are:

$$\begin{aligned}
q_x &= x \\
q_y &= y \\
p_x &= \dot{x} - \Omega y \\
p_y &= \dot{y} + \Omega x
\end{aligned}$$

Problem 2:

There are no formal solutions here. Have a look at the provided text file for ideas on how to run the programs.

<u>Problem 3</u>:

To get the Lagrangian points of a rotating potential, we need to consider the effective potential

$$\Phi_{eff} = \Phi - \frac{1}{2}\Omega^2(x^2+y^2) = \frac{1}{2}V_0^2 \ln\left(R_c^2 + x^2 + \frac{y^2}{q^2}\right) - \frac{1}{2}\Omega^2(x^2+y^2) \tag{3}$$

The Lagrangian points along the x axis ($\Rightarrow y = 0$) lie where the first derivative of the effective potential is zero, namely:

$$\frac{\partial}{\partial x}\Phi_{eff}(x, y=0) = x\left(\frac{V_0^2}{R_c^2 + x^2} - \Omega^2\right) = 0$$

which admits as solutions $x = 0$ and

$$x = \pm\sqrt{\frac{V_0^2 - \Omega^2 R_c^2}{\Omega^2}}$$

Therefore, there is always a Lagrangian point at $x = 0$ (L3) and two others (L1 and L2) if the expression for x we found is real, i.e. whenever $V_0 > \Omega R_c$.