Equilibria of collisionless systems

2rd part

Outlines

The Jeans theorems

- Symmetry and integrals of motion

Connections between barotropic fluids and ergodic stellar systems

Self-consitent spherical models with Ergodic DF

- DFs from mass distribution
    - The Eddington formula
    - Examples
- Models defined from DFs
    - Polytropes and Plummer models

Quick summary of the last lecture

Distribution function (DF)

Definition (1) $f(\vec{x}, \vec{v}, t)$ or $f(\vec{w}, t)$ such that

$f(\vec{x}, \vec{v}, t)\, d^3\vec{x}\, d^3\vec{v}$ or $f(\vec{w}, t)\, d^3\vec{w}$

is the probability that at the time t,
a randomly chosen star "i" has its position $\vec{x}_i$
an velocity $\vec{v}_i$, or phase space coordinates $\vec{w}_i$
in the ranges

$$\vec{x}_i \in [\vec{x}, \vec{x} + d\vec{x}]$$
$$\vec{v}_i \in [\vec{v}, \vec{v} + d\vec{v}]$$
$$\equiv \quad \vec{w}_i \in [\vec{w}, \vec{w} + d\vec{w}]$$

obviously:

(normalisation)

$$\boxed{\begin{aligned} &\int f(\vec{x}, \vec{v}, t)\, d^3\vec{x}\, d^3v = 1 \\ \equiv &\int f(\vec{w}, t)\, d^6\vec{w} = 1 \end{aligned}}$$

the particle is for sure somewhere in the phase space

$\boxed{f(\vec{x}, \vec{v}, t) \text{ is the probability density of the phase space.}}$

The collisionless Boltzmann equation

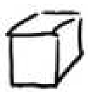
- What is the evolution of $f(\vec{w})$ over time?

As the mass, the probability is a conserved quantity. $\rho = N f$

the number of stars is a conserved quantity.

in the phase space

Continuity equation (similar than for hydrodynamics)

V  the time variation of the mass in V : $\frac{dM}{dt} = \sum_{\text{faces}} \underbrace{\rho \vec{v}}_{\text{mass flux}} \cdot d\vec{S}$

Gauss

Mass conservation

$$\underbrace{\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}_x \cdot (\rho \vec{v}) = 0}}_{\text{mass flux through the surface of the volume}}$$

Probability conservation

$$\underbrace{\boxed{\frac{\partial f}{\partial t} + \vec{\nabla}_w \cdot (f \dot{\vec{w}}) = 0}}_{\text{probability flux through the surface of the volume}}$$

The Collisionless Boltzmann equation in various coordinates

Generalized coordinates

$$\vec{p} = \frac{\partial L(\vec{q}, \vec{p})}{\partial \dot{\vec{q}}}$$

$$\boxed{\frac{\partial f}{\partial t} + \dot{\vec{q}} \cdot \frac{\partial f}{\partial \vec{q}} + \dot{\vec{p}} \cdot \frac{\partial f}{\partial \vec{p}} \equiv \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{q}} \cdot \frac{\partial H}{\partial \vec{p}} - \frac{\partial f}{\partial \vec{p}} \cdot \frac{\partial H}{\partial \vec{q}} = 0}$$

Cartesian coordinates

$$\begin{cases} p_x = \dot{x} = v_x \\ p_y = \dot{y} = v_y \\ p_z = \dot{z} = v_z \end{cases} \qquad H = \frac{1}{2}\left(v_x^2 + v_y^2 + v_z^2\right) + \Phi(x, y, z)$$

$$\boxed{\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} - \frac{\partial \Phi}{\partial \vec{x}} \cdot \frac{\partial f}{\partial \vec{v}} = 0}$$

Spherical coordinates

$$\begin{cases} p_r = \dot{r} = v_r \\ p_\theta = r^2 \dot{\theta} = r v_\theta \\ p_\phi = r^2 \sin^2(\theta \dot{\phi}) = r \sin(\theta)\, v_\phi \end{cases} \qquad H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2(\theta)}\right) + \Phi(R, \theta, \phi)$$

$$\boxed{\frac{\partial f}{\partial t} + p_r \frac{\partial f}{\partial r} + \frac{p_\theta}{r^2} \frac{\partial f}{\partial \theta} + \frac{p_\phi}{r^2 \sin^2(\theta)} \frac{\partial f}{\partial \phi} - \left(\frac{\partial \Phi}{\partial r} - \frac{p_\theta^2}{r^3} - \frac{p_\phi^2}{r^3 \sin^2(\theta)}\right) \frac{\partial f}{\partial p_r} - \left(\frac{\partial \Phi}{\partial \theta} - \frac{p_\phi^2 \cos(\theta)}{r^2 \sin^3(\theta)}\right) \frac{\partial f}{\partial p_\theta} - \frac{\partial \Phi}{\partial \phi} \frac{\partial f}{\partial p_\phi} = 0}$$

Cylindrical coordinates

$$\begin{cases} p_R = \dot{R} = v_R \\ p_\phi = R^2 \dot{\phi} = R \\ p_z = \dot{z} = v_z \end{cases} \qquad H = \frac{1}{2}\left(p_R^2 + \frac{p_\phi^2}{R^2} + p_z^2\right) + \Phi(R, \phi, z)$$

$$\boxed{\frac{\partial f}{\partial t} + p_R \frac{\partial f}{\partial R} + \frac{p_\phi}{R^2} \frac{\partial f}{\partial \phi} + p_z \frac{\partial f}{\partial z} - \left(\frac{\partial \Phi}{\partial R} - \frac{p_\phi^2}{R^3}\right) \frac{\partial f}{\partial p_R} - \frac{\partial \Phi}{\partial \phi} \frac{\partial f}{\partial p_\phi} - \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial p_z} = 0}$$

Jeans theorems

I. Any steady-state solution of the collisionless Boltzmann equation depends on the phase-space coordinates only through integrals of motion in the given potential.

Demonstration:

If a function is a solution of the steady-state collisionless Boltzmann equation, then, it is an integral of motion, thus the function depends on the phase-space coordinates only through integrals of motion (itself !).

II. Any function of integrals of motion yields a steady-state solution of the collisonless Boltzmann equation.

Extremely useful to generate DFs

Demonstration:

Assume $f(\vec{x},\vec{v}) = f(I_1(\vec{x},\vec{v}), I_2(\vec{x},\vec{v}), I_3(\vec{x},\vec{v}), ...)$ and derivate...

Equilibria of collisionless systems

Symmetries and DFs

Choices of DFs and relations with the velocity moments

$\boxed{\text{1. DFs that depend only on } H}$

(no particular symmetry except time!)

$\phi = \phi(\vec{x}, \cancel{t})$

Ergodic distribution functions

Example

$$\begin{cases} H(\vec{x}, \vec{v}) = \frac{1}{2}\vec{v}^2 + \phi(\vec{x}) \\ f = f\left(\frac{1}{2}\vec{v}^2 + \phi(\vec{x})\right) \end{cases}$$

Note: the velocity dependency is only through v^2 (isothropic)

Mean velocity

$$\bar{\vec{v}}(\vec{x}) = \frac{1}{\nu(\vec{x})} \int \vec{v}\, f\left(\frac{1}{2}\vec{v}^2 + \phi(\vec{x})\right) d^3\vec{v} = 0$$

indeed

$$\bar{v}_x(\vec{x}) = \frac{1}{\nu(\vec{x})} \int_{-\infty}^{\infty} dv_z \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_x \underbrace{v_x}_{\text{odd}} \underbrace{f\left(\frac{1}{2}\vec{v}^2 + \phi(\vec{x})\right)}_{\text{even}} = 0$$

$$\boxed{1. \text{ DFs that depend only on } \mathcal{H}}$$

Velocity dispersions

$$\sigma_{ij}^2 = \frac{1}{\nu(\bar{x})} \int \underbrace{(v_i - \underbrace{\cancel{\bar{v}_i}}_{=0})(v_j - \underbrace{\cancel{\bar{v}_j}}_{=0})} \, \delta\left(\frac{1}{2}\bar{v}^2 + \phi(\bar{x})\right) d^3\bar{v}$$

$$= \delta_{ij}\sigma^2$$

odd, except if $i=j$ $\quad (\sigma_{xx} = \sigma_{yy} = \sigma_{zz})$

$$\sigma^2 = \frac{1}{\nu(\bar{x})} \int_{-\infty}^{\infty} v_z^2 \, dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z \, \delta\left(\frac{1}{2}\bar{v}^2 + \phi(\bar{x})\right)$$

using spherical coord in velocity space : $\left\{ \begin{array}{l} dv_x\, dv_y\, dv_z = v^2 \sin\theta \, dv\, d\varphi\, d\theta \\ v_z^2 = v^2 \cos^2\theta \\ v^2 = v_x^2 + v_y^2 + v_z^2 \end{array} \right.$

$$\boxed{\sigma^2 = \frac{4}{3}\pi \frac{1}{\nu(x)} \int_0^{\infty} v^4 \, \delta\left(\frac{1}{2}v^2 + \phi(\bar{x})\right) dv}$$

$$\sigma_{ij} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$

isothropic system :

the velocity ellipsoid is a sphere

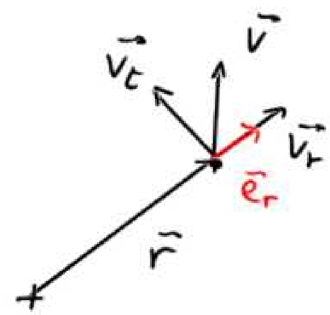
2. DFs that depend on H and $\vec{L}$

(spherical symmetry)

$\phi = \phi(r)$

We restrict our study to symmetric DFs : indep. of any direction

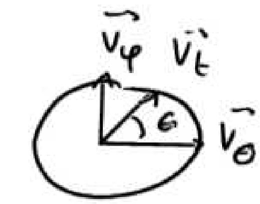
$$f(\vec{x}, \vec{v}) = f(H, L) \qquad \vec{L} \to |\vec{L}| = L$$



we consider

$v_r = \vec{v} \cdot \vec{e_r}$

radial velocity : $\vec{v_r} = v_r \vec{e_r}$



tangential velocity : $\vec{v_t} = \vec{v} - v_r \vec{e_r}$

$$\vec{v_t}^2 = \vec{v_\theta}^2 + \vec{v_\varphi}^2$$

$$V_\theta = V_t \cos\theta \qquad V_\varphi = V_t \sin\theta$$

$$dV_\theta \, dV_\varphi = V_t \, dV_t \, d\theta$$

$$(\equiv dx \, dy = r \, dr \, d\theta)$$

$$\left\{ \begin{array}{l} L = r^2 \dot{\theta} = r v_t = r\sqrt{V_\theta^2 + V_\varphi^2} \\ \\ H = \frac{1}{2} \underbrace{(V_r^2 + V_t^2)}_{V_r^2 + V_\theta^2 + V_\varphi^2} + \phi(r) \end{array} \right.$$

2. DFs that depend on H and $\vec{L}$

Mean velocity

$$\bar{v}_r = \frac{1}{\nu(x)} \int v_r \, f\left(\tfrac{1}{2}(v_r^2+v_t^2)+\phi(r), r v_t\right) d^3v$$

$$= \frac{1}{\nu(x)} \int_{-\infty}^{\infty} \underbrace{v_r}_{\text{odd in } v_r} dv_r \underbrace{\int d^2\vec{v}_t \, f\left(\tfrac{1}{2}(v_r^2+v_t^2)+\phi(r), r v_t\right)}_{\text{even in } v_r} = 0$$

$$\bar{\vec{v}}_t = \frac{1}{\nu(x)} \int \underbrace{\vec{v}_t}_{\text{odd in } v_t} d^2\vec{v}_t \underbrace{\int_{-\infty}^{\infty} dv_r \, f\left(\tfrac{1}{2}(v_r^2+v_t^2)+\phi(r), r \underbrace{v_t}_{\text{Note } v_t \in [0,\infty]}\right)}_{\text{even in } v_t} = 0$$

2. DFs that depend on H and $\vec{L}$

Velocity dispersions

veloc. in cyl. coord
$dv_\theta \, dv_\varphi \to v_t \, dv_t \, d\theta$

$$\sigma_r^2 = \frac{1}{\nu(x)} \int_{-\infty}^{\infty} v_r^2 \, dv_r \int_{-\infty}^{\infty} dv_\theta \int_{-\infty}^{\infty} dv_\varphi \, f\left(\tfrac{1}{2}(v_r^2 + v_\theta^2 + v_\varphi^2) + \phi(r), r v_t\right)$$

$$= \frac{2\pi}{\nu(x)} \int_{-\infty}^{\infty} v_r^2 \, dv_r \int_0^{\infty} dv_t \, v_t \, f\left(\tfrac{1}{2}(v_r^2 + v_t^2) + \phi(r), r v_t\right) \neq 0$$

$$\sigma_\theta^2 = \frac{1}{\nu(x)} \int_{-\infty}^{\infty} v_\theta^2 \, dv_\theta \int_{-\infty}^{\infty} dv_\varphi \int_{-\infty}^{\infty} dv_r \, f\left(\tfrac{1}{2}(v_r^2 + v_\theta^2 + v_\varphi^2) + \phi(r), r v_t\right)$$

$dv_\theta \, dv_\varphi \to v_t \, dv_t \, d\theta$

$$= \frac{1}{\nu(x)} \int_{-\infty}^{\infty} \int_0^{\infty} v_\theta^2 \, v_t \, dv_t \, dv_r \, f\left(\tfrac{1}{2}(v_r^2 + v_t^2) + \phi(r), r v_t\right)$$

$v_\theta^2 \, v_t \, dv_t = v_t^2 \cos^2\theta \, v_t \, dv_t \to \pi \, v_t^3 \, dv_t$

$$= \frac{\pi}{\nu(x)} \int_0^{\infty} dv_t \, v_t^3 \int_{-\infty}^{\infty} dv_r \, f\left(\tfrac{1}{2}(v_r^2 + v_t^2) + \phi(r), r v_t\right) \neq 0$$

integral of v_t only

2. DFs that depend on H and $\vec{L}$

Velocity dispersions

$$\sigma_\varphi^2 = \frac{1}{\nu(x)} \int_{-\infty}^{\infty} v_\varphi^2 \, dv_\varphi \int_{-\infty}^{\infty} dv_\theta \int_{-\infty}^{\infty} dv_r \, f\left(\tfrac{1}{2}(v_r^2 + v_\theta^2 + v_\varphi^2) + \phi(r), r v_t\right)$$

$dv_\theta \, dv_\varphi \to v_t \, dv_t \, d\theta$

$$= \frac{1}{\nu(x)} \int_{-\infty}^{\infty}\int_{0}^{\infty} v_\varphi^2 \; v_t \; dv_t \, dv_r \; f\left(\tfrac{1}{2}(v_r^2 + v_t^2) + \phi(r), r v_t\right)$$

$v_\varphi^2 = v_t^2 \sin^2\theta \quad \to \pi v_t^3 \, dv_t$

$$= \frac{\pi}{\nu(x)} \int_{0}^{\infty} dv_t \, v_t^3 \int_{-\infty}^{\infty} dv_r \, f\left(\tfrac{1}{2}(v_r^2 + v_t^2) + \phi(r), r v_t\right) \quad \neq 0$$

integral of v_t only

$$\boxed{\sigma_\varphi^2 = \sigma_\theta^2}$$

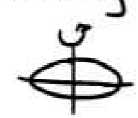
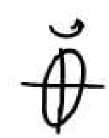
ok, spherical symmetry

$\sigma_{ij} = 0$ if $i \neq j$

Anisothropic system

$$\boxed{\sigma_r^2 \neq \sigma_\theta^2 = \sigma_\varphi^2}$$

The velocity ellipsoid is

oblate  or prolate 

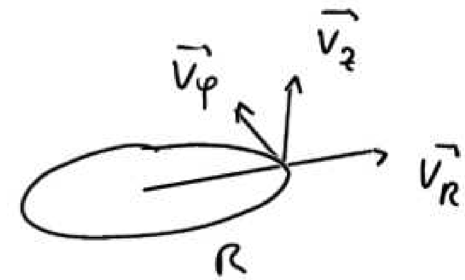
3. DFs that depend on H and L_z

(cylindrical symmetry)

$$\phi = \phi(R, |z|)$$

$$f(\vec{x}, \vec{v}) = f(H, L_z)$$

$$\begin{cases} H = \frac{1}{2}\left(v_r^2 + v_\varphi^2 + v_z^2\right) + \phi(R, z) \\ L_z = R^2 \dot{\varphi} = R v_\varphi \qquad (v_\varphi = R\dot{\varphi}) \end{cases}$$



Mean velocity

$$\bar{v}_R = \int dv_R \, v_R \int dv_z \, dv_\varphi \, f\left(\frac{1}{2}\left(v_r^2 + v_\varphi^2 + v_z^2\right) + \phi(R, z), R v_\varphi\right) = 0$$

odd in v_R

$$\bar{v}_z = \int dv_z \, v_z \int dv_R \, dv_\varphi \, f\left(\frac{1}{2}\left(v_r^2 + v_\varphi^2 + v_z^2\right) + \phi(R, z), R v_\varphi\right) = 0$$

odd in v_z

$$\bar{v}_\varphi = \int dv_\varphi \, v_\varphi \int dv_R \, dv_z \, f\left(\frac{1}{2}\left(v_r^2 + v_\varphi^2 + v_z^2\right) + \phi(R, z), R v_\varphi\right)$$

$\neq 0$ in general (net rotation)

$= 0$ only if f is an even function of $L_z = R v_\varphi$

3. DFs that depend on H and L_z

Velocity dispersions

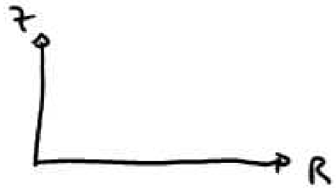
$$\sigma_R^{\,2} = \frac{1}{\nu(x)} \int dv_R \, v_R^2 \int dv_z \int dv_\varphi \, f\left(\tfrac{1}{2}\left(v_r^2 + v_\varphi^2 + v_z^2\right) + \Phi(R,z),\, R\, v_\varphi\right)$$

$$\sigma_z^{\,2} = \sigma_R^{\,2} \neq 0$$ (both variables v_R and v_z can be exchanged)

$$\sigma_\varphi^{\,2} = \frac{1}{\nu(x)} \int dv_\varphi \left(v_\varphi - \bar{v}_\varphi\right)^2 \int dv_z \, dv_R \, f\left(\tfrac{1}{2}\left(v_r^2 + v_\varphi^2 + v_z^2\right) + \Phi(R,z),\, R\, v_\varphi\right)$$

$\neq 0$

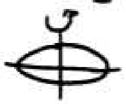
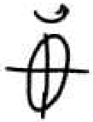
σ is isothropic in the meridional plane



Anisothropic system

$$\boxed{\sigma_\varphi^{\,2} \neq \sigma_R^{\,2} = \sigma_z^{\,2}}$$

The velocity ellipsoid is

oblate  or prolate 

Interpretation : relation between the DF and the orbits

Example 1 — 1-D potential

$$\begin{cases} E = \frac{1}{2} v^2 + \phi(x) \\ v = \pm \sqrt{2(E - \phi(x))} \end{cases}$$

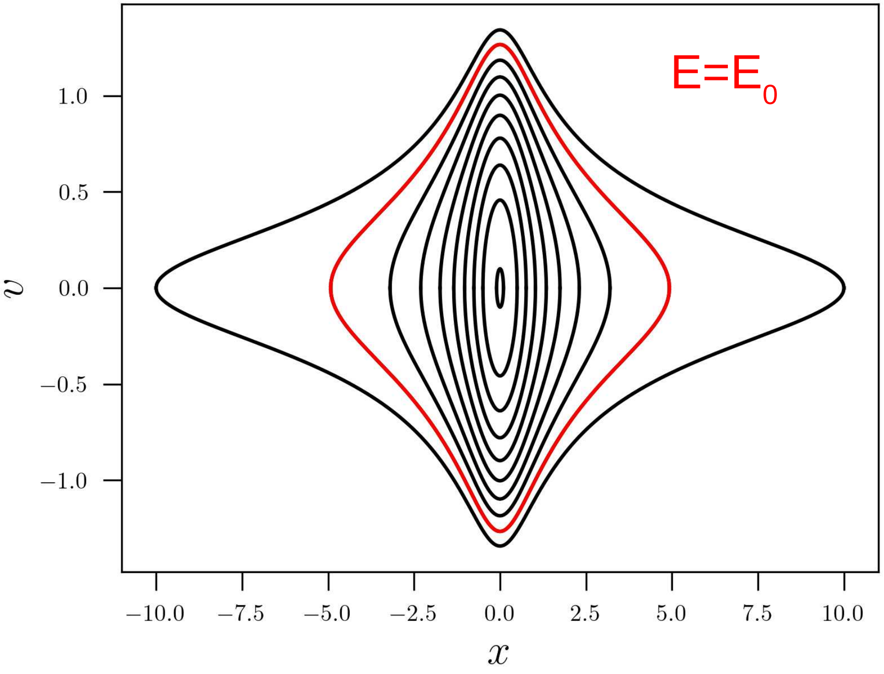
a) $f(x,v) = f(E) = \delta(E - E_0)$

$$\begin{cases} \infty & v = \pm\sqrt{2(E_0 - \phi(x))} \\ 0 & \text{instead} \end{cases}$$

b) $f(x,v) = f(E)$

↓

give a weight to orbits depending on their energy



Example 2 3D - spherical potential

- orbits described in planes, characterized by (E, L)

a) **Ergodic DF** : $f(\vec{x}, \vec{v}) = f(E)$

$$\boxed{\begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}}$$

- model built-out of all orbits of all planes with a weight that depends on the energy
(radial and circular orbits are weighted the same way) invariant under rotation (isotropic)

b) **non Ergodic DF** : $f(\vec{x}, \vec{v}) = f(E, L)$

$$\boxed{\sigma_r^2 \neq \sigma_\theta^2 = \sigma_\varphi^2}$$

- model built-out of all orbits of all planes with a weight that depends on E and L
(radial and circular orbits are weighted differently)

c) **non Ergodic DF** : $f(\vec{x}, \vec{v}) = f(E, \vec{L}) = f_E(E)\, f_L(\vec{L})$

with $f_L(\vec{L}) = 0$ if $\vec{L} \cdot \vec{e}_z \neq |\vec{L}|$

$$\boxed{\sigma_\varphi^2 \neq \sigma_R^2 = \sigma_z^2}$$

- model built-out of orbits lying in the $z=0$ plane with a weight that depends on E and L_z

no longer spherical...

Questions

Why an ergodic DF with a priori no constraint on the symmetry of the potential leads to an isotropic velocity dispersion tensor ?

$$\Phi(x, y, z) \quad f(H) \qquad \Longrightarrow \qquad \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$

Connections between barotropic fluids and ergodic stellar systems

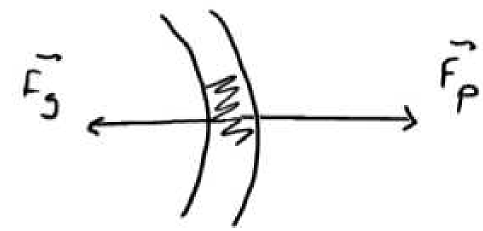
Connections between fluids and stellar systems

In fluid dynamics, the properties of a fluid at rest in a potential is obtained through the Euler equation

$$\boxed{\frac{d\vec{v}}{dt} = \underbrace{-\frac{\vec{\nabla} P}{\rho}}_{\text{pressure force}} \underbrace{- \vec{\nabla}\phi}_{\text{gravity}}}$$

<u>At rest</u> $\qquad \frac{d\vec{v}}{dt} = 0$

$$\boxed{\frac{\vec{\nabla} P}{\rho} = -\vec{\nabla}\phi}$$



<u>In 1-D</u> (isothropic case)

$$\boxed{\frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{\partial \phi}{\partial r}}$$

Equation of state (EoS) $\quad P = P(\rho, T)$

$P = P(\rho)$: barotropic (depends only on the density)

$P = K\rho^{\gamma}$: polytropic

$P = \frac{k_B T}{m}\rho$: isotherm (T = cte)

Together with the hydrostatic equation,

$$\boxed{\frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{\partial \phi}{\partial r}}$$

this relates $\underline{\rho(r) \text{ with } \phi(r)}$.

Self - gravity

The Poisson equation

$$\boxed{\vec{\nabla}^2 \phi = 4\pi G \rho}$$

This constraints the potential $\phi(r)$

or equivalently the density $\rho(r)$

Note An ergodic DF is such that the velocity dispertion is isothropic

(σ_{σ_σ}) $\equiv$ similar to a gaseous system

Idea : define a function $P(\rho)$ (an equivalent of the pressure) which is such that :

$$\boxed{\frac{\vec{\nabla} P}{\rho} = - \vec{\nabla} \phi} \qquad \frac{1}{\rho}\frac{\partial P}{\partial r} = - \frac{\partial \phi}{\partial r}$$

if spherical

If we find $P(\rho)$ for our stellar system, its density will be the same than the one of a gaseous system as the "pressure" will be equivalent.

Ergodic DF $\qquad f(\vec{x}, \vec{v}) = f\left(\frac{1}{2}\vec{v}^2 + \phi(\vec{x})\right)$

Density
$$\rho(\vec{x}) = \int d^3v \; f(\vec{x}, \vec{v})$$
$$= \int d^3v \; f\left(\frac{1}{2}\vec{v}^2 + \phi(\vec{x})\right)$$

as ρ depends on $\vec{x}$ only through ϕ, we can write:

$\rho = \rho(\phi)$ and assuming it to be bijective

$$\boxed{\phi = \phi(\rho)}$$

we can then compute $\dfrac{\partial \phi}{\partial \rho}$

Lets define the function $p(\rho)$

$$\boxed{p(\rho) = -\int_0^{\rho} d\rho' \, \rho' \, \frac{\partial \phi}{\partial \rho}(\rho')}$$

Differentiating gives

$$\frac{\partial p}{\partial \rho}(\rho) = -\rho \frac{\partial \phi}{\partial \rho}(\rho)$$

with $\rho = \rho(\vec{x})$ $\quad \frac{\partial p}{\partial \rho} = \vec{\nabla} p \cdot \frac{\partial \vec{x}}{\partial \rho}$, $\quad \frac{\partial \phi}{\partial \rho} = \vec{\nabla} \phi \cdot \frac{\partial \vec{x}}{\partial \rho}$

it becomes:

$$\boxed{\frac{\vec{\nabla} p}{\rho} = -\vec{\nabla} \phi}$$

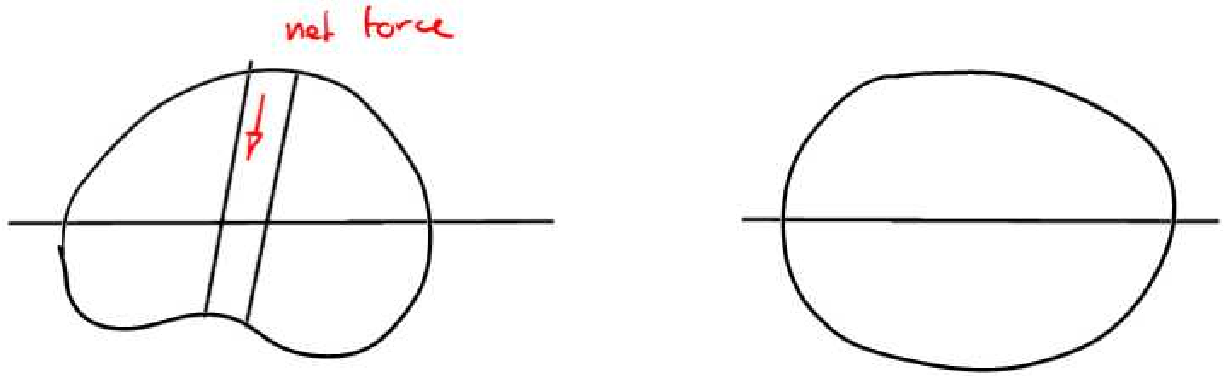
Which is the equation of equilibrium for a barotropic fluid.

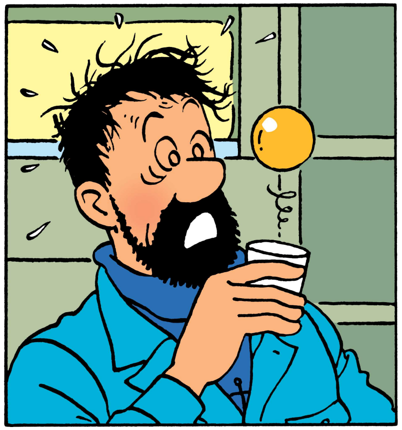
Conclusions

① To demonstrate the analogy between an ergodic stellar system and a gaseous system, it is sufficient to show that the DF leads to the same pressure form $P(\rho)$

② An ergodic isolated stellar system is spherical

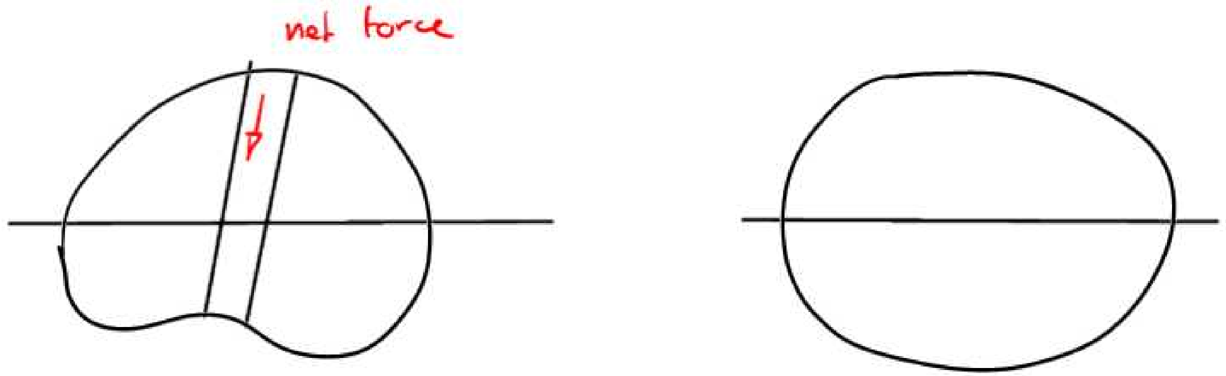
As an isolated finite, static, self-gravitating barotropic fluid must be spherical. (Lichtenstein's theorem)





As a stellar system with an ergodic DF satisfies the same equations, it must be spherical

As an isolated finite, static, self-gravitating barotropic fluid must be spherical. (Lichtenstein's theorem)



Theorem

Any isolated, finite, stellar system with an ergodic distribution function must be spherical.

Equilibria of collisionless systems

Self-consistent spherical models with ergodic DFs

Distribution function for spherical systems (Ergodic DFs)

isothropic velocity field

Goal: provide a self-consistent model for a spherical stellar system

ex: - elliptical galaxy
- galaxy cluster
- globular cluster

self-consistent = the density obtained from the DF is the one that generates the potential

i.e. is a solution of the Poisson equation

$$\boxed{\rho(\vec{x}) = N m \underbrace{\int d^3v\, f(\vec{x},\vec{v})}_{\nu(\vec{x})} \equiv \frac{1}{4\pi G} \vec{\nabla}^2 \Phi}$$

assumptions: only one type of stars (one stellar population)

i.e. all stars are modeled through the same DF.

Distribution function for spherical systems

- Method (1)

  - from $\rho(r)$ $\phi(r)$ $\rightarrow$ get $f(\varepsilon) = f(\frac{1}{2}v^2 + \phi(r))$

- Method (2)

  - assume $f(\varepsilon)$ $\rightarrow$ get $\rho(r)$

Spherical systems defined by DFs

DFs from mass distribution

Determination of the DF from the mass distribution

- We assume that $\rho(r)$ and $\phi(r)$ are known functions related together by the Poisson equation : $\nabla^2\phi = 4\pi G\rho$

- The density is related to the DF : $\nu(r) = \frac{\rho(r)}{Nm} = \frac{\rho(r)}{M}$

$$\rho(r) = M\,\nu(r) = \int f(E)\, d^3\vec{v}$$

$$E = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\dot{y}^2 + \tfrac{1}{2}\dot{z}^2 + \phi(r) = \tfrac{1}{2}v^2 + \phi(r)$$

$$= \int_0^\infty dv\, 4\pi v^2 f\left(\tfrac{1}{2}v^2 + \phi(r)\right)$$

(isothropic in the velocity space)

We are thus looking for DFs f that satisfy :

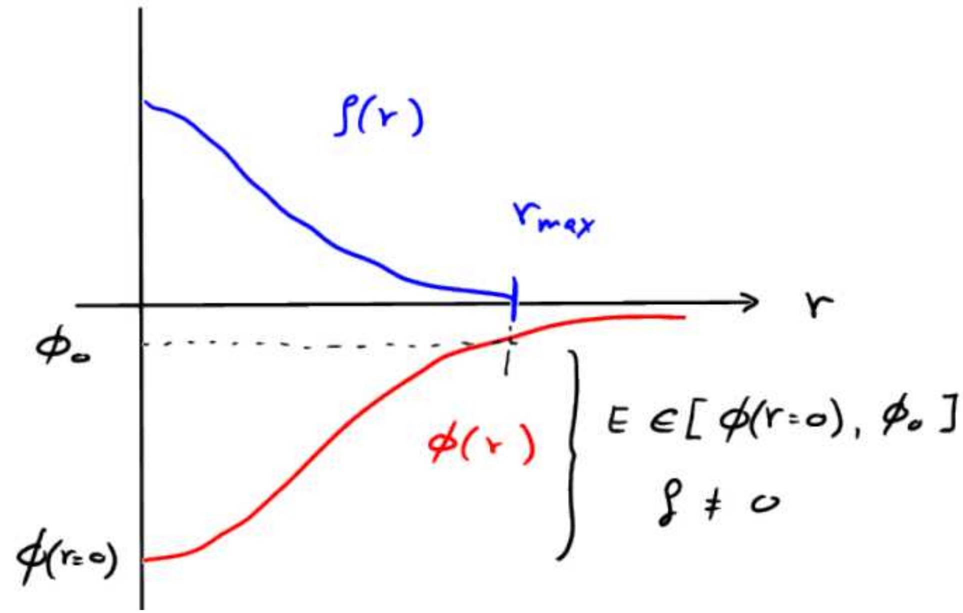
$$\boxed{\nu(r) = 4\pi\int_0^\infty dv\, v^2 f\left(\tfrac{1}{2}v^2 + \phi(r)\right)}$$

Density and potential

- $\rho(r)$ $\qquad\qquad \rho(r > r_{max}) = 0$
- $\phi(r)$ $\qquad\qquad$ no limit

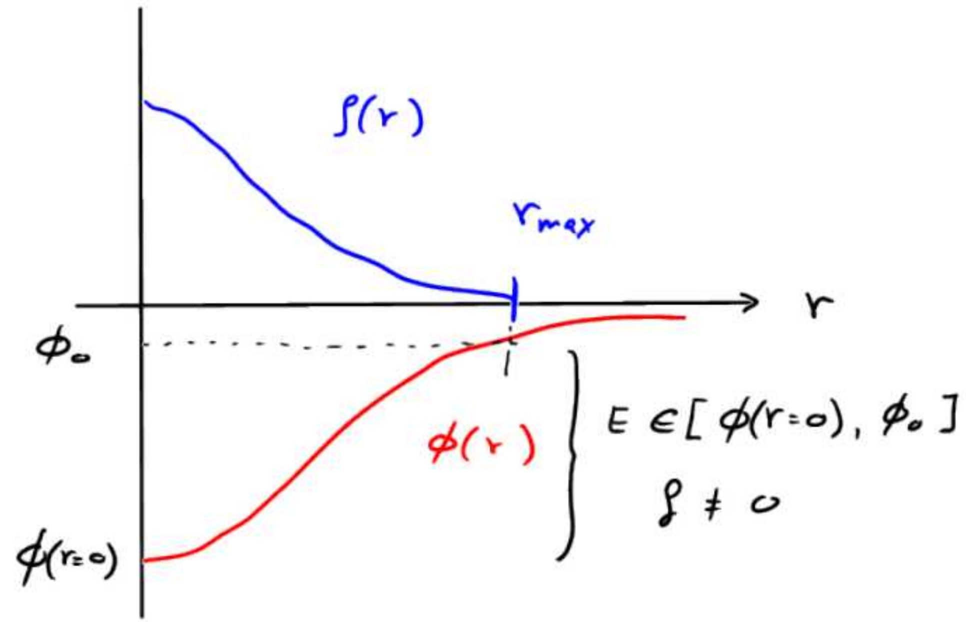
Goal: find $\rho = \rho(E)$ with

$$\rho = 0 \quad \text{if} \quad r > r_{max}$$



Density and potential

- $\rho(r)$ $\quad\quad \rho(r > r_{max}) = 0$
- $\phi(r)$ $\quad\quad$ no limit



Goal: find $\rho = \rho(E)$ with

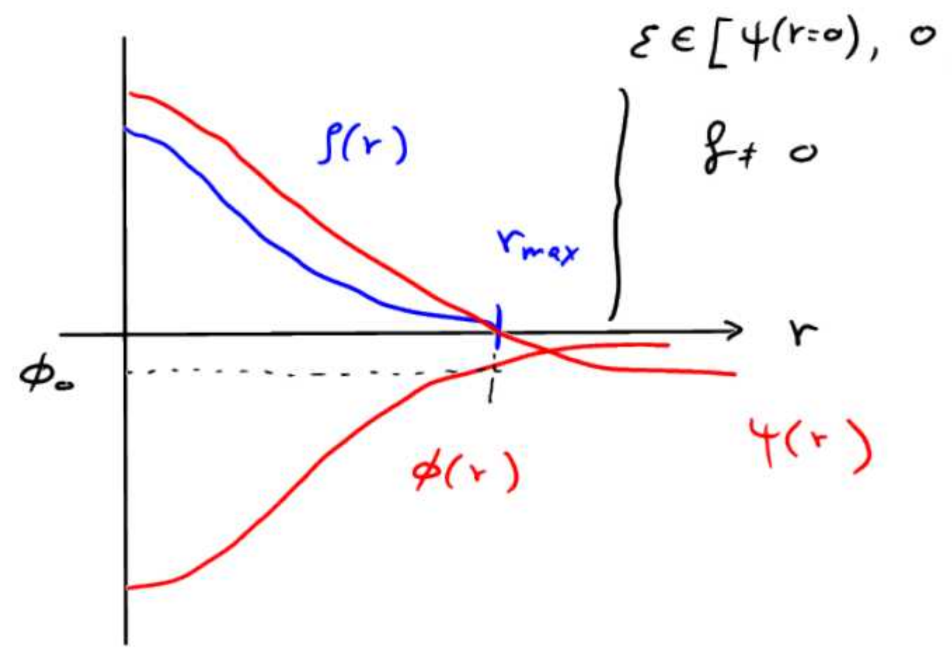
$\rho = 0$ if $r > r_{max}$

Idea new variables

relative potential

$$\begin{cases} \Psi = -(\phi - \phi_0) = -\phi + \phi_0 \\ \mathcal{E} = -(H - \phi_0) = -H + \phi_0 \end{cases}$$

relative energy $\quad\quad = \Psi - \frac{1}{2}v^2$



$$\boxed{\rho \rightarrow \rho(\mathcal{E})} \quad \begin{cases} \mathcal{E} > 0 & \rho > 0 \\ \mathcal{E} \leq 0 & \rho = 0 \end{cases}$$

Back to the density

$$\boxed{\nu(r) = 4\pi \int_0^{\infty} dV \, v^2 \, \underbrace{f\left(\tfrac{1}{2}v^2 + \phi(r)\right)}_{f(\varepsilon) \,\equiv\, f(\psi - \frac{1}{2}v^2)}}$$

<u>But</u> $f(\varepsilon) = 0$ if $\varepsilon \leq 0$ i.e $\psi - \frac{1}{2}v^2 < 0$

i.e $v > \sqrt{2\psi}$

So, we can limit the integral to:

$[0, \sqrt{2\psi}]$

$$\boxed{\nu(r) = 4\pi \int_0^{\sqrt{2\psi}} dV \, v^2 \, f\left(\psi - \tfrac{1}{2}v^2\right)}$$

Now, lets integrate over $\mathcal{E}$, rather than v

as $\quad \mathcal{E} = \psi - \frac{1}{2} v^2$

$$v = \sqrt{2(\psi - \mathcal{E})} \quad \text{and} \quad dV = \frac{-1}{\sqrt{2(\psi-\mathcal{E})}} \, d\mathcal{E}$$

$$\nu(r) = 4\pi \int_0^{\sqrt{2\psi}} dV \, v^2 \, f\left(\psi - \tfrac{1}{2} v^2\right)$$

becomes

$$\nu(r) = 4\pi \int_{\psi \, \binom{V=0}{\mathcal{E}=\psi}}^{0 \, \binom{V=\sqrt{2\psi}}{\mathcal{E}=0}} d\mathcal{E} \;\; 2(\psi - \mathcal{E}) \, f(\mathcal{E}) \, \frac{-1}{\sqrt{2(\psi-\mathcal{E})}}$$

$$= 4\pi \int_0^{\psi} d\mathcal{E} \, \sqrt{2(\psi - \mathcal{E})} \, f(\mathcal{E})$$

- if ψ is a monotonic function of r (typical potential)

$$\psi(r) \to r(\psi) \Rightarrow \nu(r) = \nu(r(\psi)) = \nu(\psi)$$

and thus

$$\frac{1}{\sqrt{8}\,\pi}\,\nu(\psi) = \int_0^{\psi} d\mathcal{E}\,\sqrt{\psi - \mathcal{E}}\,f(\mathcal{E})$$

Derivating with respect to ψ (not trivial), we get

$$\boxed{\frac{1}{\sqrt{8}\,\pi}\,\frac{\partial \nu(\psi)}{\partial \psi} = \int_0^{\psi} d\mathcal{E}\,\frac{f(\mathcal{E})}{\sqrt{\psi - \mathcal{E}}}}$$

Abel integral

Solution: Eddington's formula

$$\rho(\varepsilon) = \frac{1}{\sqrt{8}\pi^2} \frac{d}{d\varepsilon}\left[\int_0^{\varepsilon} \frac{d\psi}{\sqrt{\varepsilon - \psi}} \frac{d\nu}{d\psi} \right]$$

or

$$\rho(\varepsilon) = \frac{1}{\sqrt{8}\pi^2} \left[\int_0^{\varepsilon} \frac{d\psi}{\sqrt{\varepsilon - \psi}} \frac{d^2\nu}{d\psi^2} + \frac{1}{\sqrt{\varepsilon}} \left(\frac{d\nu}{d\psi}\right)_{\psi=0} \right]$$

Note: $\rho(\varepsilon) > 0$ only if $\int_0^{\varepsilon} \frac{d\psi}{\sqrt{\varepsilon - \psi}} \frac{d\nu}{d\psi}$

is an increasing function of ε !

How using this formula ?

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\,\pi^2} \frac{d}{d\mathcal{E}} \left[\int_0^{\mathcal{E}} \frac{d\Psi}{\sqrt{\mathcal{E}-\Psi}} \frac{d\nu}{d\Psi} \right]$$

- We start from a given $\rho(r)$, $\phi(r)$

① get r_{max} and compute $\phi_0 = \phi(r_{max})$

② a) get $\nu(r) = \rho(r)/M$

$\Psi(r) = -\phi(r) + \phi_0$

b) and $\nu = \nu(\Psi)$ if $\Psi(r)$ may be inverted

③ if $\frac{\partial \nu}{\partial \Psi}$ is analytical , compute $f(\mathcal{E})$ (Eddington's formula)

④ $f(x, v) = f(\mathcal{E}) = f(\phi_0 - \mathcal{E}) = f(\frac{1}{2}v^2 + \phi)$

Note 2a and ③ may be performed numerically

Example : Hernquist model

- $\rho(r) = \dfrac{\rho_0}{(r/a)(1 + r/a)}$

$$M(r) = 2\pi\rho_0 a^3 \frac{(r/a)^2}{(1+r/a)^2}$$

$$M = M(\infty) = 2\pi\rho_0 a^3$$

- $\phi(r) = -2\pi G\rho_0 \dfrac{a^2}{(1+r/a)}$

The density is non-zero at $r = \infty \Rightarrow \phi_0 = 0$

$$\psi(r) = -\phi(r)$$

- inverting $\phi(r)$, we have

$$r/a = \frac{2\pi G\rho_0 a^2}{\psi} - 1 = \frac{GM}{\psi a} - 1 = \frac{1}{\tilde{\psi}} - 1$$

$M = 2\pi\rho_0 a^3$

$\tilde{\psi} := \dfrac{\psi}{GM} a$

we can now express ν as $\nu(\psi)$, eliminating r/a

$$\nu(\psi) = \frac{\rho}{M} = \frac{1}{2\pi a^3}\frac{\tilde{\psi}^4}{1-\tilde{\psi}^4}$$

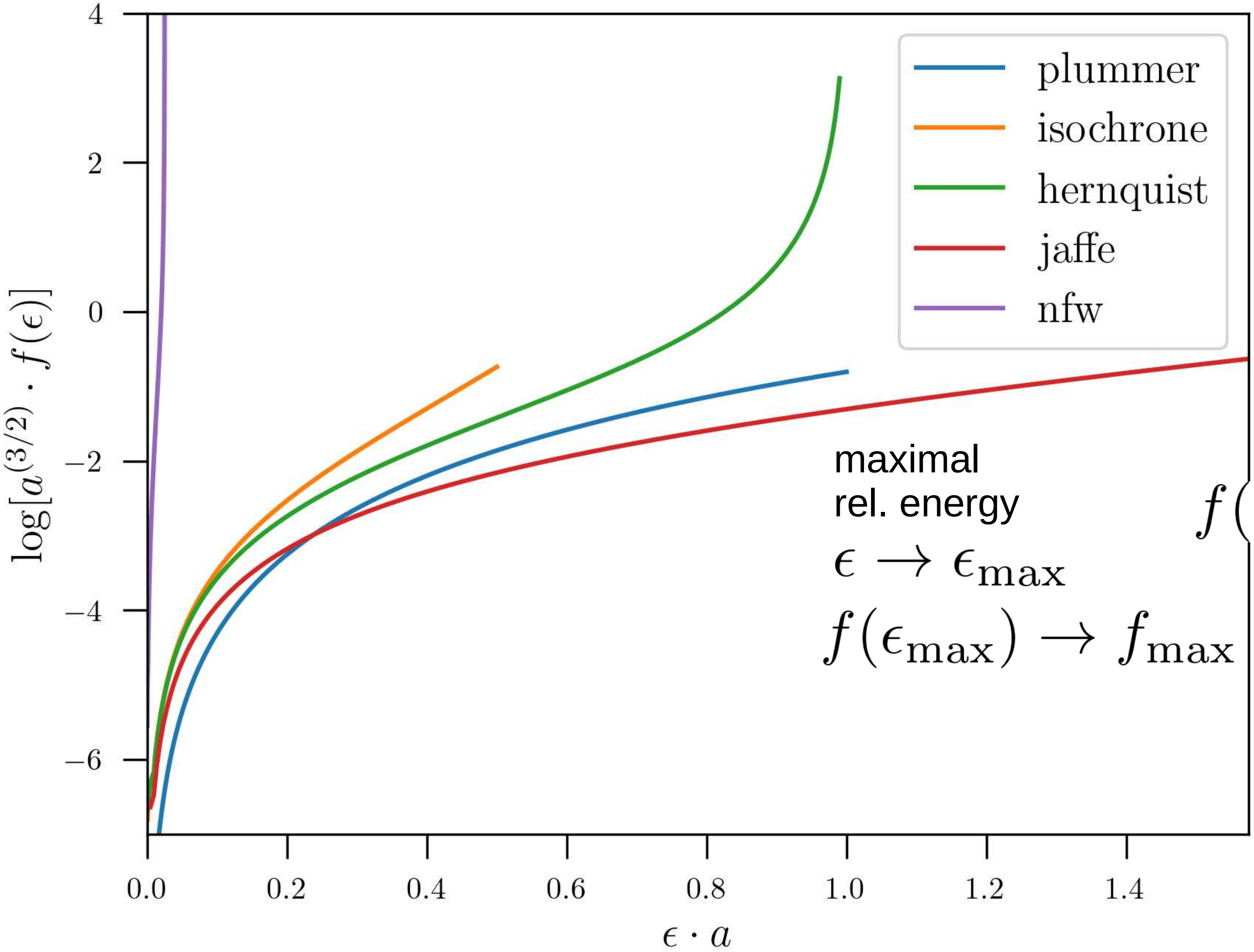
then
$$\frac{\partial \nu(\psi)}{\partial \psi} = \frac{1}{2\pi a^2 GM}\frac{\tilde{\psi}^3(4-3\tilde{\psi})}{(1-\tilde{\psi})^2}$$

And the DF becomes, using $\tilde{\mathcal{E}} = -\dfrac{\mathcal{E}a}{GM}$

$$\boxed{\begin{aligned} f(\mathcal{E}) &= \frac{\sqrt{2}}{(2\pi)^3 (GM)^2 a}\int_0^{\mathcal{E}} \frac{d\psi}{\sqrt{\mathcal{E}-\psi}}\,\frac{2\tilde{\psi}^2(6-8\tilde{\psi}+3\tilde{\psi}^2)}{(1-\tilde{\psi})^3} \\ &= \frac{1}{\sqrt{2}\,(2\pi)^3 (GMa)^{3/2}}\,\frac{\sqrt{\tilde{\mathcal{E}}}}{(1-\tilde{\mathcal{E}})^2}\left[(1-2\tilde{\mathcal{E}})(8\tilde{\mathcal{E}}^2-8\tilde{\mathcal{E}}-3) + \frac{3\arcsin(\sqrt{\tilde{\mathcal{E}}})}{\sqrt{\tilde{\mathcal{E}}(1-\tilde{\mathcal{E}})}}\right] \end{aligned}}$$

Note: Proceeding similary, it is possible to compute the DF for others spherical potentials

maximal rel. energy $\epsilon \rightarrow \epsilon_{\max}$ $f(\epsilon_{\max}) \rightarrow \infty$



maximal rel. energy $\epsilon \rightarrow \epsilon_{\max}$ $f(\epsilon_{\max}) \rightarrow f_{\max}$

no maximal rel. energy $\epsilon \rightarrow \infty$ $f(\epsilon_{\max}) \rightarrow \infty$

Plummer model

$$\Phi(r) = -\frac{GM}{\sqrt{r^2+b^2}}$$

$$\rho(r) = \frac{3M}{4\pi b^3}\left(1+\frac{r^2}{b^2}\right)^{-5/2}$$

Isochrone model

$$\Phi(r) = -\frac{GM}{b+\sqrt{r^2+b^2}}$$

$$\rho(r) = M\,\frac{3(b+\sqrt{b^2+r^2})(b^2+r^2) - r^2(b+3\sqrt{b^2+r^2})}{4\pi(b+\sqrt{b^2+r^2})^3(b^2+r^2)^{3/2}}$$

Jaffe model

$$\Phi(r) = -4\pi G\rho_0 a^2\ \ln(1+a/r)$$

$$\rho(r) = \frac{\rho_0}{(r/a)^2(1+r/a)^2}$$

Hernquist model

$$\Phi(r) = -4\pi G\rho_0 a^2\ \frac{1}{2(1+r/a)}$$

$$\rho(r) = \frac{\rho_0}{(r/a)(1+r/a)^3}$$

Equilibria of collisionless systems

Models defined from DFs

Distribution function for spherical systems

- Method 1

  - from $\rho(r)$ $\phi(r)$ $\longrightarrow$ get $f(\varepsilon) = f\left(\frac{1}{2}v^2 + \phi(r)\right)$

- Method 2

  - assume $f(\varepsilon)$ $\longrightarrow$ get $\rho(r)$

Spherical systems defined by DFs

Models defined from DFs: Polytropes

Polythropes and Plummer models

$$\boxed{f(\mathcal{E}) = \begin{cases} F\,\mathcal{E}^{\,n-3/2} & (\mathcal{E}>0) \\ 0 & (\mathcal{E}\le 0) \end{cases}}$$

F, a constant

$f = 0$ if $\mathcal{E} > 0$

$f = 0$

Corresponding density

$$\nu(r) = 4\pi \int_0^{\sqrt{2\psi}} dv\, v^2 f\left(\psi - \tfrac{1}{2}v^2\right)$$

$\times N \cdot m$

$\nu(r) \to \rho(r)$

$$\rho(r) = 4\pi F \int_0^{\sqrt{2\psi}} dv\, v^2 \left(\psi(r) - \tfrac{1}{2}v^2\right)^{n-3/2}$$

Which leads to :

$$\boxed{\rho(r) = C_n \, \psi(r)^n} \qquad (\text{for } \psi > 0)$$

relation between ρ and ϕ

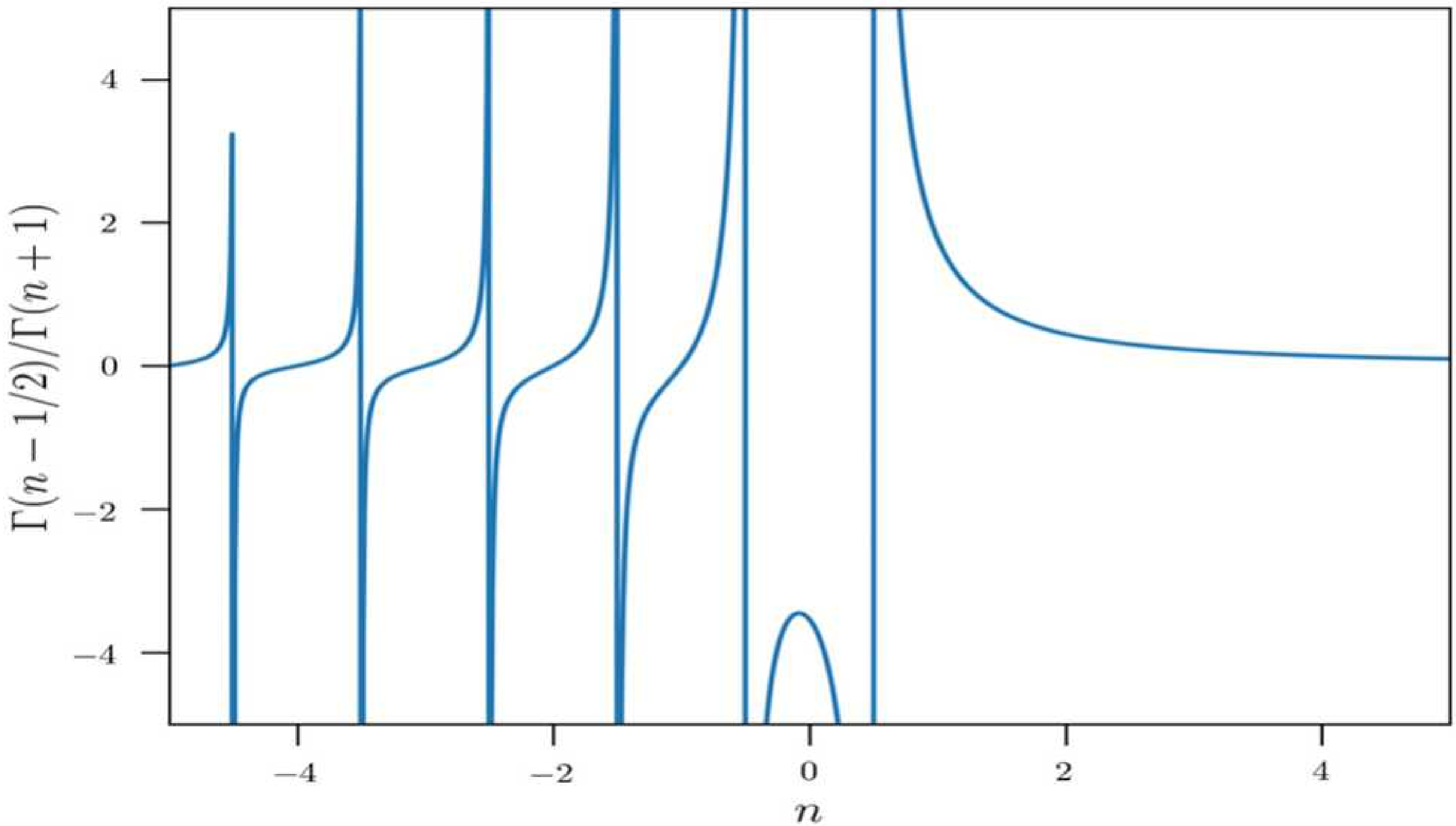
$$C_n = \frac{(2\pi)^{3/2} \left(n - \frac{3}{2}\right)! \, F}{n!} = \frac{(2\pi)^{3/2} \, \Gamma\left(n - \frac{1}{2}\right) F}{\Gamma(n+1)}$$

$$n! = \Gamma(n+1) = \int_0^\infty dt\, t^n e^{-t}$$

$$c_n \sim \frac{(n-\frac{3}{2})!}{n!} = \frac{\Gamma(n-\frac{1}{2})}{\Gamma(n+1)}$$

$n = \frac{1}{2}$

$n > \frac{1}{2}$, $c_n > 0$, $f > 0$



Demonstration

$$\rho(r) = 4\pi F \int_0^{\sqrt{2\Psi}} dv\, v^2 \left(\Psi(r) - \tfrac{1}{2}v^2\right)^{n-3/2}$$

smart substitution : introduce the variable $\theta(v)$ such that

$$v^2 = 2\Psi \cos^2\theta \quad , \quad \theta = \arccos\left(\frac{v}{\sqrt{2\Psi}}\right)$$

$$2v\,dv = -4\Psi \cos\theta \sin\theta\, d\theta$$

$$\Rightarrow \quad dv = -\frac{2\Psi \cos\theta \sin\theta\, d\theta}{\sqrt{2\Psi}\cos\theta} = -\sqrt{2\Psi}\sin\theta\, d\theta$$

$$\begin{cases} v = 0 \rightarrow \theta = \pi/2 \\ v = \sqrt{2\Psi} \rightarrow \theta = 0 \end{cases}$$

$$\Psi - \tfrac{1}{2}v^2 = \Psi - \Psi\cos^2\theta = \Psi \sin^2\theta$$

$$\rho(r) = 4\pi F \int_0^{\pi/2} \left(\sqrt{2\Psi}\sin\theta\, d\theta\right)\cdot\left(2\Psi\cos^2\theta\right)\cdot\left(\Psi \sin^2\theta\right)^{n-3/2}$$

$$= 4\pi F \int_0^{\pi/2} 2\cdot 2^{\frac{1}{2}}\, \Psi^{\frac{1}{2}}\, \Psi\, \Psi^{n-\frac{3}{2}} \cdot \cos^2\theta \quad \sin\theta^{2n-2} \quad d\theta$$

$$= 8\pi F \sqrt{2}\; \Psi^{n} \int_{0}^{\frac{\pi}{2}} \underbrace{\cos^{2}\theta}_{1-\sin^{2}} \; \sin\theta^{\,2n-2} \, d\theta$$

So, we get

$$\rho(r) = C_n \, \Psi(r)^{n} \qquad (\text{for } \Psi > 0)$$

relation between ρ and ϕ

$$C_n = \frac{(2\pi)^{3/2} \left(n-\frac{3}{2}\right)! \, F}{n!} = \frac{(2\pi)^{3/2} \, \Gamma\left(n-\frac{1}{2}\right) F}{\Gamma(n+1)}$$

Corresponding "Pressure"

$$\boxed{P(\rho) = - \int_0^{\rho} d\rho' \, \rho' \, \frac{\partial \phi}{\partial \rho}(\rho')}$$

$$\rho = C_n \psi^n$$

$$\psi = \frac{1}{C_n^{1/n}} \rho^{\frac{1}{n}} \qquad \frac{\partial \psi}{\partial \rho} = \frac{1}{C_n^{1/n}} \frac{1}{n} \rho^{\frac{1}{n}-1}$$

$$\frac{\partial \phi}{\partial \rho} = - \frac{1}{C_n^{1/n}} \frac{1}{n} \rho^{\frac{1}{n}-1}$$

$$P(\rho) = \frac{1}{C_n^{1/n}} \frac{1}{n} \int_0^{\rho} d\rho' \, \rho^{\frac{1}{n}} = \frac{1}{C_n^{1/n}} \frac{1}{n+1} \rho^{\frac{1}{n}+1}$$

$$\boxed{P(\rho) = K \rho^{\gamma}}$$

$\equiv$ Polytropic EoS

$$\begin{cases} \gamma = \frac{1}{n} + 1 \\ K = \frac{1}{C_n^{1/n}} \frac{1}{n+1} \end{cases} \qquad n = \frac{1}{\gamma - 1} \qquad C_n = \left(\frac{\gamma - 1}{K \gamma} \right)^{\frac{1}{\gamma - 1}}$$

Conclusion

The density of a stellar system described by and ergodic DF

$$f(\epsilon) \sim \epsilon^{n-3/2}$$

Is the same as a polytropic gas sphere in hydrostatic equilibrium, with:

$$P(\rho) \sim \rho^{\gamma}$$

This is why these DFs are called polytropes.

Note: from $\rho(r) = C_n \psi(r)^n$

if $\rho = \text{cte}$ $\Rightarrow$ $n = 0$

But from $C_n = \dfrac{(2\pi)^{3/2}\,\Gamma(n-\frac{1}{2})\,F}{\Gamma(n+1)}$ $\Rightarrow$ $C_n < 0$ $\rho < 0$!

① No finite ergodic stellar system is homogeneous.

② No self-gravitating homogeneous system equivalent to a self-gravitating sphere of incompressible fluid exists.

Indeed: the hydrostatic solution of an incompressible fluid of constant density requires

$$\frac{dP}{dr} = -\rho_0 \frac{d\phi}{dr} = -\frac{4}{3}\pi G \rho_0^2 r$$

not a polytropic EOS $\leftarrow$ $P = P_0 - \frac{2}{3}\pi G \rho_0^2 r^2$

Self-gravity !

$$\vec{\nabla}^2(\Phi) = 4\pi G\rho$$

The Poisson equation for spherical systems (with Ψ)

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) = -4\pi G\rho(r)}$$

$\rho = C_n \Psi^n$

$\rho^{\frac{n-1}{n}} = C_n^{\frac{n-1}{n}} \Psi^{n-1}$

With $\rho = C_n \Psi^n$ $\quad \frac{d\rho}{dr} = C_n n \Psi^{n-1}\frac{d\Psi}{dr} = C_n n \left(\frac{1}{C_n}\rho\right)^{\frac{n-1}{n}}\frac{d\Psi}{dr}$

thus $\quad \frac{d\Psi}{dr} = \frac{1}{C_n^{1/n} n}\rho^{\frac{1-n}{n}}\frac{d\rho}{dr}$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{1}{n C_n^{1/n}}\rho^{\frac{1-n}{n}}\right) + 4\pi G\rho = 0}$$

or eliminating ρ, using $\rho(r) = C_n \Psi(r)^n$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) + 4\pi G C_n \Psi^n = 0}$$

Solutions

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) + 4\pi G c_n \Psi^n = 0}$$

A. Power laws

$$\left\{\begin{array}{l}\rho(r) \sim r^{-\alpha} \\ \Psi(r) \sim r^{-\alpha/n}\end{array}\right.$$

$\rho \sim \Psi^n$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) \sim r^{-\frac{\alpha}{n}-2}$$

Poisson

$$\underbrace{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right)}_{r^{-\frac{\alpha}{n}-2}} + \underbrace{4\pi G\rho(r)}_{r^{-\alpha}} = 0$$

$$-\frac{\alpha}{n}-2 \sim -\alpha$$

$$\Rightarrow \boxed{\alpha = \frac{2n}{n-1}}$$

As the potential may not decrease faster than the Kepler potential $\frac{1}{r}$

$\left(\Psi \sim r^{-\frac{\alpha}{n}}\right)$ $\quad \underline{\frac{\alpha}{n} \leq 1} \quad \Rightarrow \quad \boxed{n \geq 3}$

B Models with finite potential and density

Define new variables $s = \frac{r}{b}$ $\quad \psi' = \frac{\psi}{\psi_0}$

where $\begin{cases} b = \left(\frac{4}{3} \pi G \psi_0^{n-1} c_n \right)^{1/2} \\ \psi_0 = \psi(0) \end{cases}$

The Poisson equation becomes $\boxed{\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\psi}{dr} \right) + 4\pi G c_n \psi^n = 0}$

$$\boxed{\frac{1}{s^2} \frac{d}{ds} \left(s^2 \frac{d\psi'}{ds} \right) = -3 \psi'^n}$$

Lane - Emden Equation

+ boundary conditions

$\begin{cases} \cdot \ \psi'(0) = 1 & \text{normalisation} \\ \cdot \ \left.\frac{d\psi'}{dr}\right)_0 = 0 & \text{no force at the center (smooth)} \end{cases}$

(In general, non trivial solutions)

Two analytical solutions $\quad n=1, \quad n=5$

$\boxed{n = 1}$

$$\boxed{\frac{1}{s^2}\frac{d}{ds}\left(s^2 \frac{d\psi'}{ds}\right) = -3\psi'}$$

linear Helmholtz Equation

$$\psi'(s) = \begin{cases} \dfrac{\sin(\sqrt{3}\, s)}{\sqrt{3}\, s} & s < \dfrac{\pi}{\sqrt{3}} \\[2ex] \dfrac{\pi}{\sqrt{3}\, s} - 1 & s \geq \dfrac{\pi}{\sqrt{3}} \end{cases}$$

UNPHYSICAL SOLUTION

$n = 1 < 3$

non physical solution

$\boxed{n = 5}$

$$\boxed{\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\psi'}{ds}\right) = -3\psi'^5}$$

consider $\psi'(s) = \dfrac{1}{\sqrt{1+s^2}}$

The Poisson Equation becomes

$$\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\psi'}{ds}\right) = -\frac{1}{s^2}\frac{d}{ds}\left(\frac{s^3}{(1+s^2)^{3/2}}\right) = -\frac{3}{(1+s^2)^{5/2}} = -3\psi'^5$$

$\rightarrow$ $\psi'(s)$ is a solution !

$$\boxed{n = 5}$$

$$\boxed{\frac{1}{s^2}\frac{d}{ds}\left(s^2 \frac{d\psi'}{ds}\right) = -3\psi'^5}$$

consider $\psi'(s) = \dfrac{1}{\sqrt{1+s^2}}$

The Poisson Equation becomes

$$\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\psi'}{ds}\right) = -\frac{1}{s^2}\frac{d}{ds}\left(\frac{s^3}{(1+s^2)^{3/2}}\right) = -\frac{3}{(1+s^2)^{5/2}} = -3\psi'^5$$

$\rightarrow$ $\psi'(s)$ is a solution !

and corresponds to the <u>Plummer model</u>

$$\boxed{\phi(r) = -\frac{GM}{\sqrt{r^2+a^2}}} \qquad \boxed{\rho(r) = \frac{3M}{4\pi a^3}\left(1 + \frac{r^2}{a^2}\right)^{-5/2}}$$

Then : what do we learn concerning the Plummer model?

We have access to its DF:

$$f(\mathcal{E}) \begin{cases} \sim \mathcal{E}^{n-3/2} \sim \left(\dfrac{GM}{\sqrt{r^2+a^2}} - \dfrac{1}{2} v^2 \right)^{7/2} \\ = 0 \quad \text{if} \quad \dfrac{GM}{\sqrt{r^2+a^2}} - \dfrac{1}{2} v^2 < 0 \end{cases}$$

We have access to the kinematics structure:

① Velocity distribution function

$$P_r(v) = \frac{f(\frac{1}{2}v^2 + \phi(r))}{\nu(r)} \sim \underbrace{\left(1+\frac{r^2}{a^2}\right)^{5/2}}_{\frac{1}{\rho}} \underbrace{\left(\frac{GM}{\sqrt{r^2+a^2}} - \frac{1}{2} v^2 \right)^{7/2}}_{\mathcal{E}^{7/2}}$$

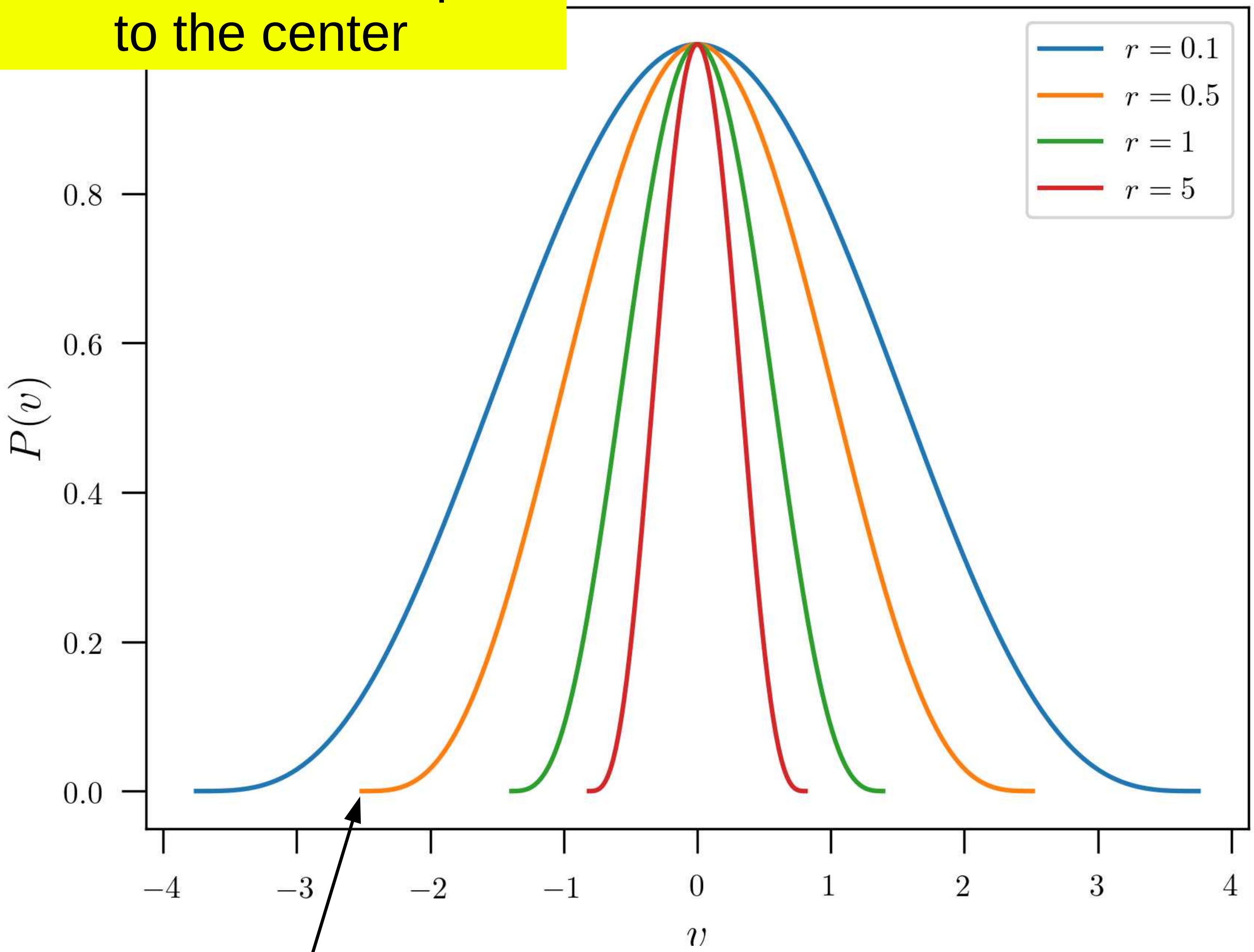
② Velocity dispersion

$$\sigma^2 = 4\pi \frac{1}{\nu(x)} \int_0^{v_{max} = \sqrt{2\psi}} v^4 f\left(\tfrac{1}{2}v^2 + \phi(\vec{x})\right) dv$$

$$= 4\pi \frac{1}{\nu(r)} \int_0^{v_{max}} v^4 \left(\frac{1}{2} v^2 - \frac{GM}{\sqrt{r^2+a^2}} \right)^{7/2} dv$$

The Plummer velocity distribution function

Normalized with respect to the center



$$\forall r, \exists\, v_{\max}\, \text{such that}\, \epsilon > 0 \Rightarrow f = 0$$

The End