Astrophysics III: Stellar and galactic dynamics

Exercises

Problem 1:

Show that the collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \tag{1}$$

in
a) spherical coordinates is

$$\frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial r} + \frac{v_\theta}{r}\frac{\partial f}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial f}{\partial \phi} + \left[\frac{v_\theta^2 + v_\phi^2}{r} - \frac{\partial \Phi}{\partial r}\right]\frac{\partial f}{\partial v_r} + \\ + \left[\frac{1}{r}\left(\frac{v_\phi^2}{\tan\theta} - v_r v_\theta\right)\right]\frac{\partial f}{\partial v_\theta} - \frac{v_\phi}{r}\left(v_r + \frac{v_\theta}{\tan\theta}\right)\frac{\partial f}{\partial v_\phi} = 0 \tag{2}$$

with $\Phi = \Phi(r)$.
b) in cylindrical coordinates is

$$\frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial r} + \frac{v_\phi}{r}\frac{\partial f}{\partial \phi} + v_z \frac{\partial f}{\partial z} + \left[\frac{v_\phi^2}{r} - \frac{\partial \Phi}{\partial r}\right]\frac{\partial f}{\partial v_r} - \frac{v_r v_\phi}{r}\frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0 \tag{3}$$

with $\Phi = \Phi(r, z)$.

Hints:
The gradients in spherical and cylindrical coordinates are given by

$$\nabla_{sph} f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi$$
$$\nabla_{cyl} f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi + \frac{\partial f}{\partial z}\mathbf{e}_z$$

To obtain expressions for the $\mathbf{a} = \dot{\mathbf{v}} = (\dot{v}_r, \dot{v}_\theta, \dot{v}_\phi)$ and $\dot{\mathbf{v}} = (\dot{v}_r, \dot{v}_\theta, \dot{v}_z)$, use the Euler-Lagrange equations of motion to find expressions for v_r, v_θ, v_ϕ, v_z first.