

# Astrophysics III: Stellar and galactic dynamics

## Solutions

**Problem 1:****a) Spherical Coordinates:**

Inserting the gradient expression into the Boltzmann equation in Cartesian coordinates and writing the scalar products explicitly gives:

$$\frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial r} + \frac{v_\theta}{r} \frac{\partial f}{\partial \theta} + \frac{v_\phi}{r \sin \phi} \frac{\partial f}{\partial \phi} + \dot{v}_r \frac{\partial f}{\partial v_r} + \dot{v}_\theta \frac{\partial f}{\partial v_\theta} + \dot{v}_\phi \frac{\partial f}{\partial v_\phi} = 0 \quad (1)$$

We have

$$v_r = \dot{r}, \quad v_\theta = r\dot{\theta}, \quad v_\phi = r \sin \theta \dot{\phi}$$

and the Lagrangian

$$L = \frac{1}{2} \left( \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) - \Phi(r)$$

To obtain the Equations of Motion, we use

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

For  $q_i = r$ :

$$\begin{aligned} \frac{d}{dt} \dot{r} - r\dot{\theta}^2 - r \sin^2 \theta \dot{\phi}^2 + \frac{\partial \Phi}{\partial r} &= 0 \\ \dot{v}_r &= \frac{v_\theta^2 + v_\phi^2}{r} - \frac{\partial \Phi}{\partial r} \end{aligned}$$

For  $q_i = \theta$ :

$$\begin{aligned} \frac{d}{dt} (r^2 \dot{\theta}) - r^2 \sin \theta \cos \theta \dot{\phi}^2 &= 0 = \frac{d}{dt} (r \cdot (r\dot{\theta})) - (r \sin \theta \dot{\phi})^2 \frac{\cos \theta}{\sin \theta} \\ \frac{d}{dt} (r\dot{\theta}) &= \frac{1}{r} \left( \frac{(r \sin \theta \dot{\phi})^2}{\tan \theta} - \dot{r}(r\dot{\theta}) \right) \\ \dot{v}_\theta &= \frac{1}{r} \left( \frac{v_\phi^2}{\tan \theta} - v_r v_\theta \right) \end{aligned}$$

For  $q_i = \phi$ :

$$\begin{aligned}\frac{d}{dt} \left( r^2 \sin^2 \theta \dot{\phi} \right) &= 0 = \frac{d}{dt} \left( r \sin \theta \cdot (r \sin \theta \dot{\phi}) \right) \\ \frac{d}{dt} \left( r \sin \theta \dot{\phi} \right) &= -(r \sin \theta \dot{\phi}) \left( \frac{\dot{r}}{r} + \frac{r \dot{\theta}}{r \tan \theta} \right) \\ \dot{v}_\phi &= -\frac{v_\phi}{r} \left( v_r + \frac{v_\theta}{\tan \theta} \right)\end{aligned}$$

Inserting  $\dot{v}_r$ ,  $\dot{v}_\theta$ ,  $\dot{v}_\phi$  into expression (1) gives the required result.

b) Cylindrical Coordinates:

Inserting the gradient expression into the Boltzmann equation in Cartesian coordinates and writing the scalar products explicitly gives:

$$\frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial r} + \frac{v_\phi}{r} \frac{\partial f}{\partial \phi} + v_z \frac{\partial f}{\partial z} + \dot{v}_r \frac{\partial f}{\partial v_r} + \dot{v}_\phi \frac{\partial f}{\partial v_\phi} + \dot{v}_z \frac{\partial f}{\partial v_z} = 0 \quad (2)$$

We have

$$v_r = \dot{r}, \quad v_\phi = r\dot{\phi}, \quad v_z = \dot{z}$$

and the Lagrangian

$$L = \frac{1}{2} \left( \dot{r}^2 + r^2 \dot{\phi}^2 + \dot{z}^2 \right) - \Phi(r, z)$$

To obtain the Equations of Motion, we use

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

For  $q_i = r$ :

$$\begin{aligned}\frac{d}{dt} \dot{r} - r \dot{\phi}^2 + \frac{\partial \Phi}{\partial r} &= 0 \\ \dot{v}_r &= \frac{v_\phi^2}{r} - \frac{\partial \Phi}{\partial r}\end{aligned}$$

For  $q_i = \phi$ :

$$\begin{aligned}\frac{d}{dt} \left( r^2 \dot{\phi} \right) - 0 &= 0 = \frac{d}{dt} \left( r \cdot (r \dot{\phi}) \right) \\ \frac{d}{dt} \left( r \dot{\phi} \right) &= -\frac{1}{r} \dot{r} \cdot r \dot{\phi} \\ \dot{v}_\phi &= -\frac{v_r v_\phi}{r}\end{aligned}$$

For  $q_i = z$ :

$$\begin{aligned}\frac{d}{dt} \dot{z} + \frac{\partial \Phi}{\partial z} &= 0 \\ \dot{v}_z &= -\frac{\partial \Phi}{\partial z}\end{aligned}$$

Inserting  $\dot{v}_r$ ,  $\dot{v}_\phi$ ,  $\dot{v}_z$  into expression (2) gives the required result.