Quantum Information Processing

Solution Homework 10

Exercise 1 *Product states and CSHS inequality*

a) The possible outcomes are the following four cases:

- 1)
$$|\alpha\rangle \otimes |\beta\rangle, \quad a = 1, b = 1, \quad p(1, 1|\alpha, \beta) = |\langle\alpha|\varphi_A\rangle|^2 |\langle\beta|\varphi_B\rangle|^2$$
Considering the measurement only in Alice's lab: $p_A(1|\alpha) = |\langle\alpha|\varphi_A\rangle|^2$. Considering the measurement only in Bob's lab: $p_B(1|\beta) = |\langle\beta|\varphi_B\rangle|^2$.
- 2)
$$|\alpha\rangle \otimes |\beta_\perp\rangle, \quad a = 1, b = -1, \quad p(1, -1|\alpha, \beta) = |\langle\alpha|\varphi_A\rangle|^2 |\langle\beta_\perp|\varphi_B\rangle|^2$$
Considering the measurement only in Alice's lab: $p_A(1|\alpha) = |\langle\alpha|\varphi_A\rangle|^2$. Considering the measurement only in Bob's lab: $p_B(-1|\beta) = |\langle\beta_\perp|\varphi_B\rangle|^2$.
- 3)
$$|\alpha_\perp\rangle \otimes |\beta\rangle, \quad a = -1, b = 1, \quad p(-1, 1|\alpha, \beta) = |\langle\alpha_\perp|\varphi_A\rangle|^2 |\langle\beta|\varphi_B\rangle|^2$$
Considering the measurement only in Alice's lab: $p_A(-1|\alpha_\perp) = |\langle\alpha_\perp|\varphi_A\rangle|^2$. Considering the measurement only in Bob's lab: $p_B(1|\beta) = |\langle\beta|\varphi_B\rangle|^2$.
- 4)
$$|\alpha_\perp\rangle \otimes |\beta_\perp\rangle, \quad a = -1, b = -1, \quad p(-1, -1|\alpha, \beta) = |\langle\alpha_\perp|\varphi_A\rangle|^2 |\langle\beta_\perp|\varphi_B\rangle|^2$$
Considering the measurement only in Alice's lab: $p_A(-1|\alpha_\perp) = |\langle\alpha_\perp|\varphi_A\rangle|^2$. Considering the measurement only in Bob's lab: $p_B(-1|\beta) = |\langle\beta_\perp|\varphi_B\rangle|^2$.

b) Since the locality assumption is satisfied as shown above i.e $p(a, b|\alpha, \beta) = p_A(a|\alpha)p_B(b|\beta)$, as well as for all other choices of angles, we can proceed as with the analysis of hidden variable theories to prove that $|X| \leq 2$ (here there is no hidden variable or if you wish the distribution is $q(\lambda) = \delta(\lambda)$ the delta distribution at $\lambda = 0$).

Exercise 2 *The difference between a Bell state and a statistical mixture of* $|00\rangle$, $|11\rangle$

a) For the Bell state the density matrix is simply

$$\rho_{\text{Bell}} = |B_{00}\rangle\langle B_{00}| = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$$

In array form

$$\rho_{\text{Bell}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Note this is a rank one matrix as it should since ρ_{Bell} is a rank one projector with eigenvalues 1 and $0, 0, 0$. We also check $Tr\rho_{\text{Bell}} = 1$.

b) For the statistical mixture we have

$$\rho_{\text{stat}} = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$$

In array form

$$\rho_{\text{stat}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note this is a rank two matrix as it should since ρ_{Bell} is a rank one projector with eigenvalues $1, 0, 0, 1$. We also check $Tr\rho_{\text{stat}} = 1$.

c) In the Bell state the average of the observable $\mathcal{B}$ is

$$Tr(\mathcal{B}\rho_{\text{Bell}}) = Tr(\mathcal{B}|B_{00}\rangle\langle B_{00}|) = Tr\langle B_{00}|\mathcal{B}|B_{00}\rangle = \langle B_{00}|\mathcal{B}|B_{00}\rangle$$

The expression as a function of angles is calulated in the course

$$\cos 2(\alpha - \beta) + \cos 2(\alpha - \beta') - \cos 2(\alpha' - \beta) + \cos 2(\alpha' - \beta')$$

and for the optimal choice of angles the values is $2\sqrt{2}$.
In the statistical state we have by linearity and cyclicity of the trace

$$Tr(\mathcal{B}\rho_{\text{stat}}) = \frac{1}{2}\langle 00|\mathcal{B}|00\rangle + \frac{1}{2}\langle 11|\mathcal{B}|11\rangle$$

For $A \otimes B$ we get the contribution

$$\frac{1}{2}\langle 0|A|0\rangle\langle 0|B|0\rangle + \frac{1}{2}\langle 1|A|1\rangle\langle 1|B|1\rangle = (\cos^2\alpha - \sin^2\alpha)(\cos^2\beta - \sin^2\beta) = \cos 2\alpha \cos 2\beta$$

So for the correlation coefficient we have

$$Tr(\mathcal{B}\rho_{\text{stat}}) = \cos 2\alpha \cos 2\beta + \cos 2\alpha \cos 2\beta' - \cos 2\alpha' \cos 2\beta + \cos 2\alpha' \cos 2\beta'$$

For the optimal angles of CSHS we find $\sqrt{2}$. Note that it is possible to prove this expression can never be greater than 2.

Exercise 3 *Ekert 1991 protocol*

a) When Alice and Bob use the same basis i.e., $(\alpha = 0, \beta'' = 0)$ or $(\alpha'' = -\frac{\pi}{8}, \beta' = -\frac{\pi}{8})$, the measurement outcome is the same on both sides. So they get common bits $a = b''$ or $a'' = b'$. This happens on average $2N/9$ times.

b) Alice and Bob perform their sets of N measurements each. They keep the outcomes secret. After measurements are finished they reveal publicly the choices of basis. They retain for the one-time pad only the bits corresponding to the same basis choices. The average length of the one time pad is then $2N/9$.

c) For the security test Alice and Bob take all events when the basis choices are the 4 Bell/CSHS choices involving angles $\alpha, \alpha', \beta, \beta'$ and compute the correlation coefficient. If there is no eavesdropper they should find $2\sqrt{2}$ (in an ideal noiseless situation).

d) The possible values of γ are the α's and $\alpha_\perp$'s (so 6 possible values). Similarly for δ the possible values are β's and $\beta_\perp$'s (so 6 possible values).

Since the eaves dropper leaves the state in a product state from the first exercise it follows that $-2 \leq X \leq 2$. This is seprated by a sizable gap from $2\sqrt{2}$ so the eavesdropper is detected.