Exercise 9.1. Show that a covector field ξ on a smooth manifold M is smooth if and only if for any smooth vector field X on M the function $\langle \xi, X\rangle : M \to \mathbb{R}$ defined by $\langle \xi, X\rangle(p) = \xi_p(X_p)$ is smooth.

Solution. Suppose ξ is a smooth covector field and X is a smooth vector field. Let us show that $\langle \xi, X\rangle$ is a smooth function. Take any chart (U, φ) of M. Then we can write $\xi|_U = \sum_i \xi_i \,\mathrm{d}\varphi^i$, and $X = \sum_j X^j \frac{\partial}{\partial \varphi^j}$, where $\xi_i,\ X^j : U \to \mathbb{R}$ are smooth functions. Then the function

$$\langle \xi, X\rangle = \sum_i \xi_i \, X^i$$

is a smooth function on U since the product and sum of smooth functions is smooth.

Viceversa, now suppose ξ is a covector field such that $\langle \xi, X\rangle$ is a smooth function for every smooth vector field X on M. Using bump functions we can show that this is also true for a vector field X defined on an open set $U \subseteq M$: the function $\langle \xi|_U, X\rangle : U \to \mathbb{R}$ is smooth in this case as well.

Proof. To see that $\langle \xi|_U, X\rangle$ is smooth at a point $p \in U$, we summon a bump function η supported on U that is $\equiv 1$ in an open neighborhood W of p. Then we define a smooth vector field $Y \in \mathfrak{X} M$ by setting $Y|_U \equiv \eta X$ and $Y|_{M \setminus \operatorname{supp} \eta} \equiv 0$. This field Y coincides with X on W, therefore the function $\langle \xi, X\rangle$ coincides with the smooth function $\langle \xi, Y\rangle$ on W. This proves that $\langle \xi, X\rangle$ is smooth at the point p. □

Let (U, φ) a smooth chart of M. The component functions of ξ with respect to φ, are the functions $\xi_i : U \to \mathbb{R}$ such that

$$\xi|_U = \sum_i \xi_i \,\mathrm{d}\varphi^i.$$

This functions can be computed by the formula $\xi_i = \langle \xi, \frac{\partial}{\partial \varphi^i}\rangle$, thus they are are smooth. This shows that ξ is smooth on U. The same reasoning shows that ξ is smooth everywhere. □

Exercise 9.2 (Properties of the differential)**.** Let $f, g \in C^\infty(M, \mathbb{R})$.

(a) Prove the formulas: $\mathrm{d}(af + bg) = a\,\mathrm{d}f + b\,\mathrm{d}g$ (where a, b are constants), $\mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f$, $\mathrm{d}\left(\frac{f}{g}\right) = \frac{g\,\mathrm{d}f - f\,\mathrm{d}g}{g^2}$ (on the set where $g \neq 0$)

Solution. Here we use the fundamental properties of tangent vectors, namely the linearity and the Leibniz rule. For every vector field $X \in TM$ we have

$$\mathrm{d}(af + bg)(X) = X(a\,f + b\,g) = a\,X(f) + b\,X(g) = a\ \mathrm{d}f(X) + b\ \mathrm{d}g(X)$$

and

$$\mathrm{d}(fg)(X) = X(fg) = f\,X(g) + g\,X(f) = f\ \mathrm{d}g(X) + g\ \mathrm{d}f(X)$$

Recall that if $h : M \to \mathbb{R}$ is constant then $X(h) = 0$ for every vector field $X \in TM$, therefore

$$0 = X(g/g) = g\,X(1/g) + \frac{1}{g} X(g)$$

which lead us to $X(1/g) = -X(g)/g^2$. Hence we obtain

$$\mathrm{d}(f/g)(X) = X(f/g) = f\,X(1/g) + \frac{1}{g}\,X(f) = \frac{g\,X(f) - f\,X(g)}{g^2} = \frac{g\ \mathrm{d}f - f\ \mathrm{d}g}{g^2}(X)$$

□

(b) If $h : \mathbb{R} \to \mathbb{R}$ is a smooth function then $\mathrm{d}(h \circ f) = (h' \circ f)\,\mathrm{d}f$.

Solution. This is a consequence of the chain rule. Given $p \in M$, let (U, x^i) a smooth chart centered at p. Then let us write the local representation for $\mathrm{d}(h \circ f)$:

$$\mathrm{d}(h \circ f)|_p = \sum_i \left.\frac{\partial(h \circ f)}{\partial x^i}\right|_p \mathrm{d}x^i|_p$$

The standard chain rule says that $\left.\frac{\partial(h\circ f)}{\partial x^i}\right|_p = h'(f(p)) \left.\frac{\partial f}{\partial x^i}\right|_p$, hence

$$\mathrm{d}(h \circ f)_p = h'(f(p)) \sum_i \left.\frac{\partial f}{\partial x^i}\right|_p \mathrm{d}x^i|_p = h'(f(p))\,\mathrm{d}f|_p.$$

□

(c) If $\mathrm{d}f \equiv 0$, then f is constant on each connected component of M.

Solution. Let $p \in M$. Take a chart (U, ϕ) defined at p whose domain $U \subseteq M$ is connected, and let $\widetilde{f} = f \circ \phi^{-1} \in \mathcal{C}^\infty(\widetilde{U})$ be the local expression of f. Then we have

$$\mathrm{d}f|_p = \sum_i \left.\frac{\partial f}{\partial \phi^i}\right|_p \mathrm{d}\phi^i|_p = \sum_i \partial_i \widetilde{f}(\phi(p))\,\mathrm{d}\phi^i|_p \quad \text{for all points } p \in U.$$

Thus if $\mathrm{d}f \equiv 0$, then all the partial derivatives of the function $\widetilde{f} : \widetilde{U} \to \mathbb{R}$ vanish on $\widetilde{U}$. Since $\widetilde{U}$ is connected, we see by elementary calculus that $\widetilde{f}$ is constant on $\widetilde{U}$, therefore f is constant on U. This proves that if $\mathrm{d}f \equiv 0$, then f is locally constant on M. Therefore f is constant on each connected component of M. □

Exercise 9.3 (Closed and exact 1-forms)**.** Let M be a smooth manifold, $\omega \in \Omega^1(M)$.

(a) Show that for every $p \in M$ there exists $f \in C^\infty(M)$ such that $\omega|_p = \mathrm{d}f|_p$.

Note that this is only an equality of the covectors at one single point p.

Solution. Fix $p \in M$, and let (U, ϕ) be a local chart. Writing ω and $\mathrm{d}f$ in coordinates yields

$$\omega|_p = \sum_i a_i \ \mathrm{d}\phi^i|_p, \qquad \text{and} \qquad \mathrm{d}f|_p = \sum_i \left.\frac{\partial f}{\partial \phi^i}\right|_p \mathrm{d}\phi^i|_p$$

for some real numbers a_i. Then define a smooth function $g = \sum_i a_i \phi^i$. Clearly $\left.\frac{\partial g}{\partial \phi^i}\right|_p = a_i$ and so $\mathrm{d}g|_p = \omega_p$. To obtain a function defined on the whole manifold M, we use a bump function $\eta \in C^\infty(M)$ that is 1 in a neighborhood of p and has support in U. Then the function $f : M \to \mathbb{R}$ defined as $g \cdot \eta$ on U and 0 outside $\operatorname{supp}\eta$ is smooth and satisfies $\mathrm{d}f|_p = \mathrm{d}g|_p = \omega|_p$ since differentials act locally. □

(b) Write $\xi = \sum_i \xi_i\,\mathrm{d}\phi^i$ in some chart (U, ϕ). Show that if ξ is exact, then

$$\frac{\partial}{\partial \phi^j}\xi_i = \frac{\partial}{\partial \phi^i}\xi_j \quad \text{on } U. \tag{1}$$

Solution. Suppose ξ is exact, i.e., $\xi = \mathrm{d}f$ for some smooth function $f : M \to \mathbb{R}$. The local expression $\widetilde{f} = f \circ \phi^{-1}$ is a smooth function on $\widetilde{U} = \phi(U) \subseteq \mathbb{R}^n$. Thus by Schwarz's theorem on the symmetry of second derivatives we have

$$\frac{\partial^2 \widetilde{f}}{\partial x^i \partial x^j} = \frac{\partial^2 \widetilde{f}}{\partial x^j \partial x^i} \quad \text{on } \widetilde{U}$$

for all indices i, j. We thus obtain the following identity for f:

$$\frac{\partial}{\partial \phi^i}\frac{\partial}{\partial \phi^j} f = \frac{\partial}{\partial \phi^j}\frac{\partial}{\partial \phi^i} f \quad \text{on } U.$$

Now, the components of ξ w.r.t. the chart ϕ are $\xi_i = \xi(\frac{\partial}{\partial \phi^i}) = \mathrm{d}f(\frac{\partial}{\partial \phi^i}) = \frac{\partial}{\partial \phi^i} f$. Thus the identity that we proved is the same as (1). ☐

(c) Use the preceding fact to write down a 1-form which is not exact.

Solution. A simple example is to define the following 1-form on $\mathbb{R}^2$:

$$\omega = y\,\mathrm{d}x - x\;\mathrm{d}y$$

where (x, y) are the standard coordinates. Then the component functions are

$$\omega_0 = y, \quad \omega_1 = -x$$

and so

$$\frac{\partial \omega_0}{\partial y} = 1 \neq -1 = \frac{\partial \omega_1}{\partial x}.$$

☐

Remark: A 1-form that satisfies (1) *for all charts* (U, ϕ) *is called* ***closed****. We have just proved that closedness is a necessary condition for exactness. However, it is not always sufficient. The topology of* M *comes into play: e.g. on a convex subset of* $\mathbb{R}^n$ *any closed 1-form is exact. But on the punctured plane* $\mathbb{R}^2 \setminus \{0\}$ *we can construct a closed 1-form that is not exact.*

Exercise 9.4 (A closed 1-form that is not exact)**.** Let $M = \mathbb{R}^2 \backslash \{0\}$. Let $\omega \in \Omega^1(M)$ be given by

$$\omega = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2}.$$

Compute the integral of ω along the curve

$$\gamma : [0, 2\pi] \to M : t \mapsto (\cos t, \sin t).$$

Conclude that ω is not exact.

Solution. Recall that the line integral of ω along the curve gamma is defined as

$$\int_\gamma \omega = \int_0^{2\pi} \omega_{\gamma(t)}(\gamma'(t))\;\mathrm{d}t$$

Since $\omega_{\gamma(t)}(\gamma'(t)) = \cos^2 t + \sin^2 t = 1$ then $\int_\gamma \omega = 2\pi$. The fundamental theorem for line integrals implies that the integral of an exact 1-form over a closed curve is zero, hence ω is not an exact 1-form.

Remark: Notice that the 1-form ω is closed since

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2 + y^2}\right) = -\frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right)$$

☐

Exercise 9.5. Let (x, y) be the standard coordinates on $\mathbb{R}^2$ and let (r, φ) be the polar coordinates.

(a) Express $\mathrm{d}x$ and $\mathrm{d}y$ in terms of $\mathrm{d}r$ and $\mathrm{d}\varphi$ (wherever the latter are defined).

Solution. Let $(x, y) = (r \cos \phi, r \sin \phi)$ be the standard polar coordinate transformation. We have $\mathrm{d}x = \mathrm{d}(r \cos \phi) = \cos \phi\;\mathrm{d}r - r \sin \phi\;\mathrm{d}\phi$ and $\mathrm{d}y = \mathrm{d}(r \sin \phi) = \sin \phi\;\mathrm{d}r + r \cos \phi\;\mathrm{d}\phi$. ☐

(b) Let $G : \mathbb{R}^2 \to \mathbb{R}$, $G(x, y) = x^2 + y^2$. Let t be the standard coordinate on $\mathbb{R}$. Compute $G^*(\mathrm{d}t)$.

Solution. $G^*(\mathrm{d}t) = \mathrm{d}G = 2x\;\mathrm{d}x + 2y\;\mathrm{d}y$ ☐

Exercise 9.6 (Line integrals)**.** .

(a) Let M be a smooth manifold, $\gamma : I = [a, b] \to M$ a smooth curve and let $\xi \in \Omega^1(M)$. Denote by t the standard coordinate on $\mathbb{R}$. Show that $\int_\gamma \xi = \int_I \gamma^* \xi$.

Solution. We have that $\gamma^*\theta$ is a one-form on $[a,b]$ and since $\Omega^1(\mathbb{R})$ has the global frame $\mathrm{d}t$ there exists $f \in C^\infty([a,b])$ such that $\gamma^*\theta = f\,\mathrm{d}t$. In fact, the function f is given by

$$f(t) = \gamma^*\theta(\frac{\partial}{\partial t}) = \theta|_{\gamma(t)}(\gamma'(t)).$$

Hence

$$\int_\gamma \theta = \int_a^b \theta_{\gamma(t)}(\gamma'(t))\ \mathrm{d}t = \int_a^b f(t)\ \mathrm{d}t$$

□

(b) (Change of variables for 1-forms) Show that if $\sigma : I \to J$ is a positive (i.e. order preserving) diffeo between two intervals $I = [a,b]$, $J = [c,d]$, then $\int_I \sigma^*\theta = \int_J \theta$ for any 1-form $\theta \in \Omega^1(J)$.

Hint: Compute the derivatives of the functions $F(s) = \int_a^s \sigma^*\theta$ and $G(t) = \int_c^t \theta$.

What happens if σ is a negative (i.e. order reversing) diffeo ?

Solution. We write $\theta = g\,\mathrm{d}y$, $\sigma^*\theta = f\,\mathrm{d}x$. We can compute f in terms of g as follows:

$$\begin{aligned} f(s) = \sigma^*\theta\left(\frac{\partial}{\partial s}\right) &= \theta\left(\sigma_*\frac{\partial}{\partial s}\right) \\ &= \theta\left(\sigma'(s)\cdot\frac{\partial}{\partial s}\right) = \sigma'(s)\cdot\theta\left(\frac{\partial}{\partial s}\right) = \sigma'(s)\cdot g(\sigma(s)). \end{aligned}$$

Now we consider the functions $F(s) = \int_a^s \sigma^*\theta$ and $G(t) = \int_c^t \theta$. By the fundamental theorem of integral calculus we have $F'(s) = f(s)$ and $G'(t) = g(t)$.

We claim that $F(s) = G(\sigma(s))$ for all $s = [a,b]$. Indeed, both functions F and $G\circ\sigma$ have value 0 at $s = a$, and their derivatives coincide: $(G\circ\sigma)'(s) = G'(\sigma(s))\cdot\sigma'(s) = f(s) = F'(s)$.

We conclude that $F(b) = \int_a^b \sigma^*\theta$ equals $G(\sigma(b)) = G(d) = \int_c^d \theta$.

Now consider the case that $\sigma : I \to J$ is an order-reversing diffeomorphism. To keep having $\sigma(a) = c$ and $\sigma(b) = d$ we write $I = [a,b]$ and $J = [d,c]$. In this case the same argument as above proves that the integral $\int_I \sigma^*\theta := \int_a^b f$ is equal to the integral $\int_c^d g = -\int_d^c g = -\int_J \theta$. Therefore $\int_I \sigma^*\theta = -\int_J \theta$. □

(c) (Reparametrization invariance of curve integrals) If two $\mathcal{C}^1$ curves $\gamma : J \to M$, $\beta : I \to M$ are equivalent as oriented curves, in the sense that β is a positive reparametrization of γ (i.e. $\beta = \gamma\circ\sigma$, where $\sigma : I \to J$ is a positive diffeo), then $\int_\gamma \xi = \int_\beta \xi$ for any 1-form $\xi \in \Omega^1(M)$. Prove this using the definition via pullback.

Solution.

$$\begin{aligned} \int_\beta \xi = \int_I \beta^*\xi &= \int_I (\gamma\circ\sigma)^*\xi \\ &= \int_I \sigma^*(\gamma^*\xi) = \int_J \gamma^*\xi = \int_\gamma \xi \end{aligned}$$

□