Equilibria of collisionless systems

3rd part

Outlines

Models defined from Dfs

- Polytropic models
- The isothermal sphere

Anisotropic distribution function in spherical systems

- Motivation
- General concepts
- Example of an anisotropic DF
- Application to the Hernquist model

The Jeans Equations

- Motivations
- The Jeans Equations and conservation laws
- The Jeans Equations in Spherical coordinates
- The Jeans Equations in Cylindrical coordinates

Distribution function for spherical systems

- Method (1)

  - from $\rho(r)$ $\phi(r)$ $\rightarrow$ get $f(\varepsilon) \equiv f\left(\frac{1}{2}v^2 + \phi(r)\right)$

- Method (2)

  - assume $f(\varepsilon)$ $\rightarrow$ get $\rho(r)$

Spherical systems defined by DFs

Models defined from DFs: Polytropes

Polythropes and Plummer models

$$f(\mathcal{E}) = \begin{cases} F\,\mathcal{E}^{n-3/2} & (\mathcal{E} > 0) \\ 0 & (\mathcal{E} \leq 0) \end{cases}$$

F, a constant

$f = 0$ if $\mathcal{E} > 0$

$\rho = 0$

Which leads to :

$$\boxed{\rho(r) = c_n \psi(r)^n} \qquad (\text{for } \psi > 0)$$

relation between ρ and ϕ

$$c_n = \frac{(2\pi)^{3/2}\left(n-\frac{3}{2}\right)!\,F}{n!} = \frac{(2\pi)^{3/2}\,\Gamma\left(n-\frac{1}{2}\right)F}{\Gamma(n+1)}$$

Self-gravity !

$$\vec{\nabla}^2(\Phi) = 4\pi G\rho$$

The Poisson equation for spherical systems (with ψ)

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = -4\pi G\rho(r)}$$

$\rho = C_n \psi^n$

$\rho^{\frac{n-1}{n}} = C_n^{\frac{n-1}{n}} \psi^{n-1}$

With $\rho = C_n \psi^n$ $\qquad \dfrac{d\rho}{dr} = C_n n \psi^{n-1}\dfrac{d\psi}{dr} = C_n n \left(\dfrac{1}{C_n}\rho\right)^{\frac{n-1}{n}}\dfrac{d\psi}{dr}$

thus $\qquad \dfrac{d\psi}{dr} = \dfrac{1}{C_n^{1/n}\, n}\rho^{\frac{1-n}{n}}\dfrac{d\rho}{dr}$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{1}{n C_n^{1/n}}\rho^{\frac{1-n}{n}}\right) + 4\pi G\rho = 0}$$

or eliminating ρ, using $\rho(r) = C_n \psi(r)^n$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) + 4\pi G C_n \psi^n = 0}$$

Solutions

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) + 4\pi G c_n \Psi^n = 0}$$

A. Power laws

$$\left\{\begin{array}{l} \rho(r) \sim r^{-\alpha} \\ \Psi(r) \sim r^{-\alpha/n} \end{array}\right. \qquad \rho \sim \Psi^n$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) \sim r^{-\frac{\alpha}{n}-2}$$

Poisson

$$\underbrace{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right)}_{r^{-\frac{\alpha}{n}-2}} + \underbrace{4\pi G \rho(r)}_{r^{-\alpha}} = 0$$

$$-\frac{\alpha}{n}-2 \sim -\alpha$$

$$\Rightarrow \quad \boxed{\alpha = \frac{2n}{n-1}}$$

As the potential may not decrease faster than the Kepler potential $\frac{1}{r}$

$\left(\Psi \sim r^{-\frac{\alpha}{n}}\right)$ $\qquad \frac{\alpha}{n} \leq 1 \quad \Rightarrow \quad \boxed{n \geq 3}$

B Models with finite potential and density

Define new variables $s = \frac{r}{b}$ $\quad \psi' = \frac{\psi}{\psi_0}$

where
$$\begin{cases} b = \left(\frac{4}{3}\pi G \psi_0^{n-1} c_n\right)^{1/2} \\ \psi_0 = \psi(0) \end{cases}$$

The Poisson equation becomes $\quad \boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\psi}{dr}\right) + 4\pi G c_n \psi^n = 0}$

$$\boxed{\frac{1}{s^2}\frac{d}{ds}\left(s^2 \frac{d\psi'}{ds}\right) = -3\psi'^n}$$

+ boundary conditions

$$\begin{cases} \bullet\ \psi'(0) = 1 & \text{normalisation} \\ \bullet\ \left.\frac{d\psi'}{dr}\right)_0 = 0 & \text{no force at the center (smooth)} \end{cases}$$

Lane - Emden Equation

(In general, non trivial solutions)

$\boxed{n = 5}$

$$\boxed{\frac{1}{s^2}\frac{d}{ds}\left(s^2 \frac{d\psi'}{ds}\right) = -3\psi'^5}$$

consider $\psi'(s) = \dfrac{1}{\sqrt{1+s^2}}$

The Poisson Equation becomes

$$\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\psi'}{ds}\right) = -\frac{1}{s^2}\frac{d}{ds}\left(\frac{s^3}{(1+s^2)^{3/2}}\right) = -\frac{3}{(1+s^2)^{5/2}} = -3\psi'^5$$

$\rightarrow$ $\psi'(s)$ is a solution !

$\boxed{n = 5}$ $\boxed{\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\psi'}{ds}\right) = -3\psi'^5}$

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$\rightarrow$ $\psi'(s)$ is a solution !

and corresponds to the <u>Plummer model</u>

$$\boxed{\phi(r) = -\frac{GM}{\sqrt{r^2+a^2}}} \qquad \boxed{\rho(r) = \frac{3M}{4\pi a^3}\left(1+\frac{r^2}{a^2}\right)^{-5/2}}$$

Then : what do we learn concerning the Plummer model ?

We have access to its DF :

$$f(\mathcal{E}) \begin{cases} \sim \mathcal{E}^{n-3/2} \sim \left(\frac{GM}{\sqrt{r^2+a^2}} - \frac{1}{2} v^2 \right)^{7/2} \\ = 0 \quad \text{if} \quad \frac{GM}{\sqrt{r^2+a^2}} - \frac{1}{2} v^2 < 0 \end{cases}$$

We have access to the kinematics structure :

① Velocity distribution function

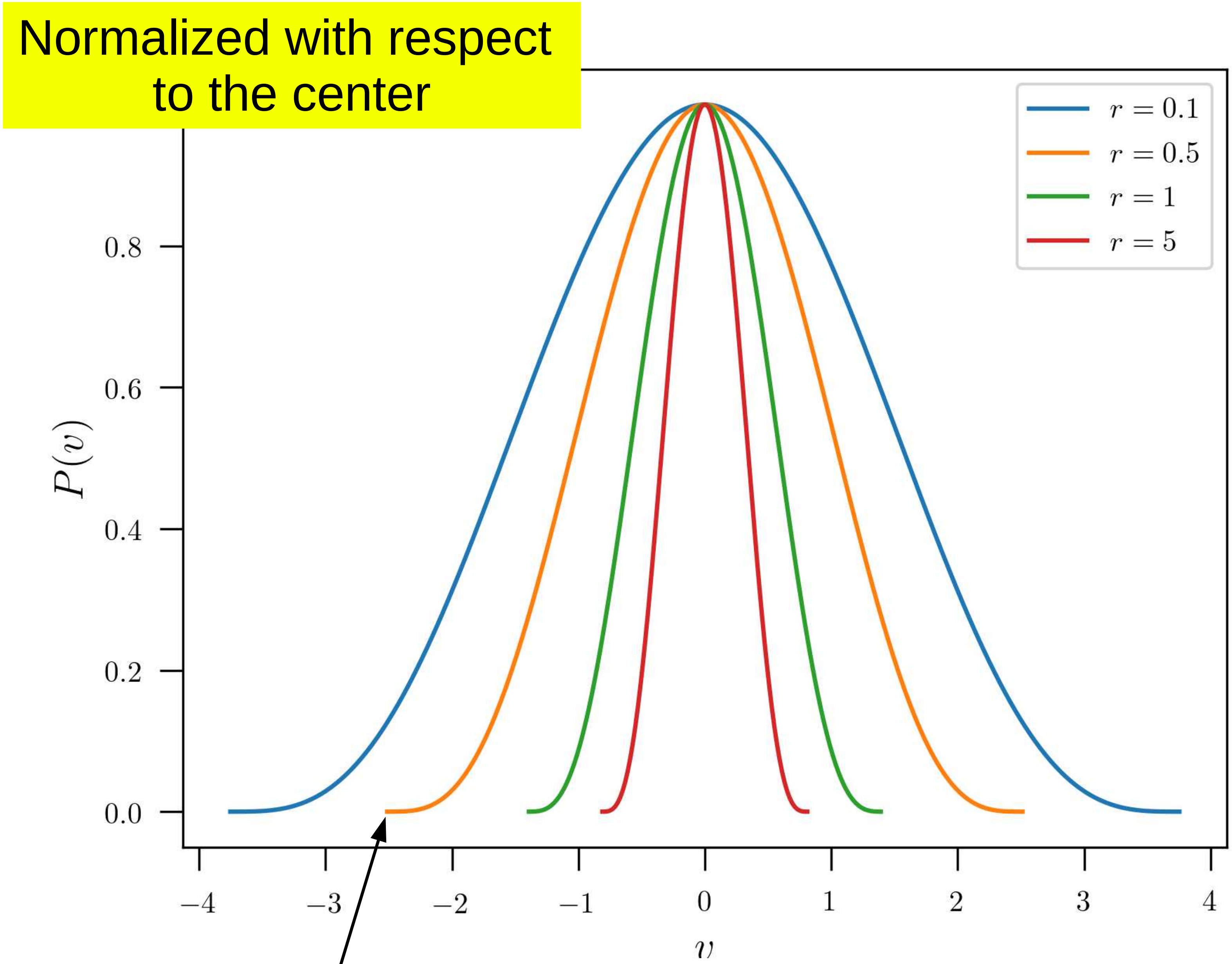
$$P_r(v) = \frac{f(\frac{1}{2} v^2 + \phi(r))}{\nu(r)} \sim \begin{cases} \underbrace{\frac{\left(1+\frac{r^2}{a^2}\right)^{5/2}}{\frac{1}{\rho}}}\underbrace{\left(\frac{GM}{\sqrt{r^2+a^2}} - \frac{1}{2} v^2 \right)^{7/2}}_{\mathcal{E}^{7/2}} & \mathcal{E} > 0 \\ 0 & \mathcal{E} \leq 0 \end{cases}$$

② Velocity dispersion

$$\sigma^2 = 4\pi \frac{1}{\nu(x)} \int_0^{v_{max} = \sqrt{2\psi}} v^4 f\left(\frac{1}{2} v^2 + \phi(\vec{x})\right) dv$$

$$= 4\pi \frac{1}{\nu(r)} \int_0^{v_{max}} v^4 \left(\frac{1}{2} v^2 - \frac{GM}{\sqrt{r^2+a^2}} \right)^{7/2} dv$$

The Plummer velocity distribution function

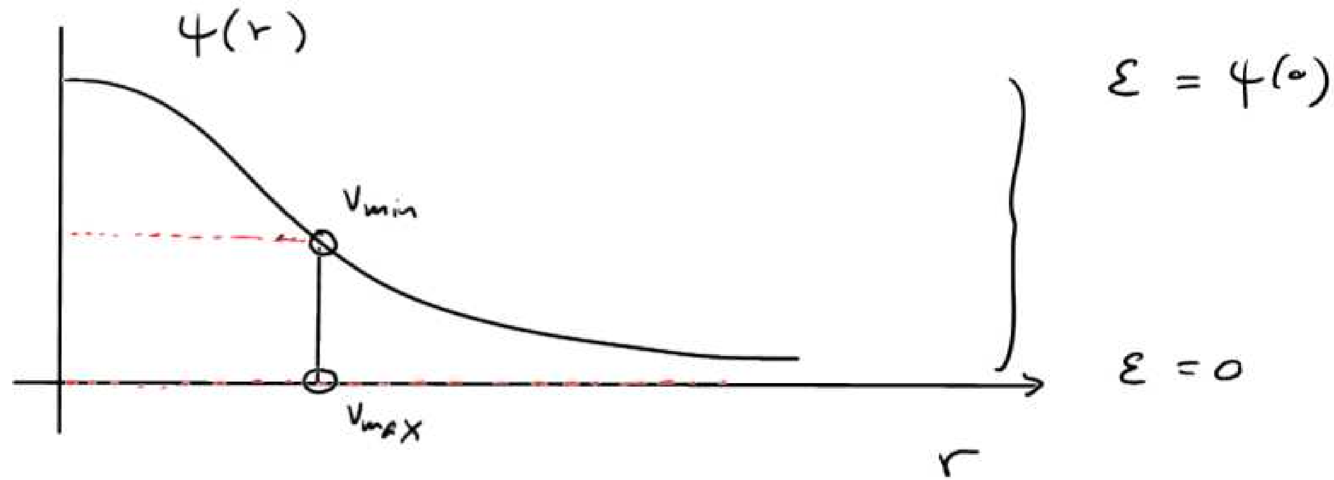


$$\forall r, \exists\, v_{\max}\, \text{such that}\, \epsilon > 0 \Rightarrow f = 0$$

Interpretation

$$f_r(v) = \begin{cases} \left(\dfrac{GM}{\sqrt{r^2+a^2}} - \dfrac{1}{2} v^2 \right)^{7/2} & \mathcal{E} > 0 \\ 0 & \mathcal{E} \leq 0 \end{cases}$$

$$\mathcal{E} = \psi - \frac{1}{2} v^2$$



in r, the minimum velocity is $V_{min} = 0$

orbits with $r_{max} = 0$, $V(r_{max}) = 0$

the maximum velocity is $V_{max} = \sqrt{2\psi(r)}$

orbits with $\mathcal{E} = 0$ $(r_{max} = \infty)$

Models defined from DFs: Isothermal spheres

Stellar system with the DF (Isothermal)

$$\boxed{f(\varepsilon) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}}\, e^{\frac{\varepsilon}{\sigma^2}}}$$

with $\varepsilon = \Psi - \frac{1}{2}v^2$

$$\rho(r) = 4\pi \int_0^{\infty} v^2 \frac{\rho_1}{(2\pi\sigma^2)^{3/2}}\, e^{\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}} = \rho_1\, e^{\frac{\Psi}{\sigma^2}}$$

$$\left(\int_0^{\infty} \frac{v^2 \; e^{\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}}}{(2\pi\sigma^2)^{3/2}}\, dv = \frac{e^{\frac{\Psi}{\sigma^2}}}{4\pi} \right)$$

$$\boxed{\rho(r) = \rho_1\, e^{\frac{\Psi}{\sigma^2}}}$$

$$\boxed{\rho(\Psi) = \rho_1\, e^{\frac{\Psi}{\sigma^2}}}$$

"Pressure" $\quad P(\rho) = \int_0^{\rho} d\rho' \rho' \frac{\partial \phi}{\partial \rho'} \equiv - \int_0^{\rho} d\rho' \rho' \frac{\partial \psi}{\partial \rho'}$

Derivating $\rho(\psi) = \rho_1 \, e^{\frac{\psi}{\sigma^2}}$ with respect to ρ

$$\frac{\partial \rho}{\partial \rho} = 1 = \rho_1 \, e^{\frac{\psi}{\sigma^2}} \frac{1}{\sigma^2} \frac{\partial \psi}{\partial \rho} = \frac{1}{\sigma^2} \rho \frac{\partial \psi}{\partial \rho}$$

$\Rightarrow \quad \rho \frac{\partial \psi}{\partial \rho} = \sigma^2$ and $\boxed{P(\rho) = \sigma^2 \rho}$

Isothermal EoS

$$\sigma^2 \equiv \frac{k_B T}{m}$$

The structure of an isothermal self-gravitating sphere of gas with an EoS

$$\boxed{P(\rho) = \frac{k_B T}{m} \rho}$$

is identical to the one of a collisionless self-gravitating system with a DF

$$\boxed{f(\mathcal{E}) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} e^{\frac{\mathcal{E}}{\sigma^2}}} \qquad \text{if} \quad \sigma^2 \equiv \frac{k_B T}{m}$$

wich leads to $\quad P(\rho) = \sigma^2 \rho$

Velocity distribution function

- collisionless isothermal sphere

$$P_r(v) = \frac{f(\varepsilon)}{\nu(\varepsilon)} \sim \frac{e^{\frac{1}{\sigma^2}\left(-\frac{1}{2}v^2+\psi(r)\right)}}{e^{\frac{1}{\sigma^2}\psi}} \sim e^{-\frac{v^2}{2\sigma^2}}$$

similar

- Gas sphere : (elastic collisions between particles)

$\Rightarrow$ Maxwell - Bolzman distribution $P_r(v) \sim e^{-\frac{mv^2}{2k_BT}} \equiv e^{-\frac{v^2}{2\sigma^2}}$

Note The correspondance between gaseous polythrope and stellar collisionless systems is not always as close a for the isothermal sphere

- gaseous polytrope : σ is allways Maxwellian and isothrope
- stellar system : σ given by f is no necessarily Maxwellian and may be anisothrope (if not ergodic)

Velocity dispersion

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \frac{1}{\nu} \int d^3v \; v^2 \, \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} \, e^{\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}}$$

spherical coord
in vel. space

$$= \frac{\frac{4}{3}\pi \int_0^\infty v^4 \, e^{\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}} dv}{4\pi \int_0^\infty v^2 \, e^{\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}} dv} = \frac{2\sigma^2}{3} \frac{\int_0^\infty dx \, x^4 e^{-x^2}}{\int_0^\infty dx \, x^2 e^{-x^2}} = \sigma^2$$

$$-x^2 = \frac{\Psi - \frac{1}{2}v^2}{\sigma^2}$$

σ^2 is indep. of r

What is the corresponding density / potential

$\rho(r)$, $\phi(r)$ of the system ?

Self-gravity !

$$\vec{\nabla}^2(\Phi) = 4\pi G\rho$$

The Poisson Equation

$$\frac{1}{r}\frac{d}{dr}\left(r^2 \frac{d\Psi}{dr}\right) = -4\pi G \rho(r)$$

$$\ln\rho = \ln\rho_1 + \frac{\Psi}{\sigma^2}$$

$$\frac{d\ln\rho}{dr} = \frac{1}{\sigma^2}\frac{d\Psi}{dr}$$

yields

$$\boxed{\frac{d}{dr}\left(r^2 \frac{d\ln\rho}{dr}\right) = -\frac{4\pi G}{\sigma^2} r^2 \rho(r)}$$

Solutions of the Poisson equation

$$\boxed{\frac{d}{dr}\left(r^2 \frac{d\ln\rho}{dr}\right) = -\frac{4\pi G}{\sigma^2} r^2 \rho(r)}$$

A. Power law

$$\rho \sim r^{-b}$$

Poisson $\Rightarrow$ $\quad -b = -\frac{4\pi G}{\sigma^2} r^{2-b}$

$$b = 2$$

$$\boxed{\rho(r) = \frac{\sigma^2}{2\pi G r^2}}$$

Singular isothermal sphere

Notes

1. The specific energy (σ^2) is constant everywhere
2. The velocity dispersion is isothropic

Maximal equilibrium ?

But ρ and ϕ diverges at $r=0$!
$M(r)$ diverges at $r = \infty$

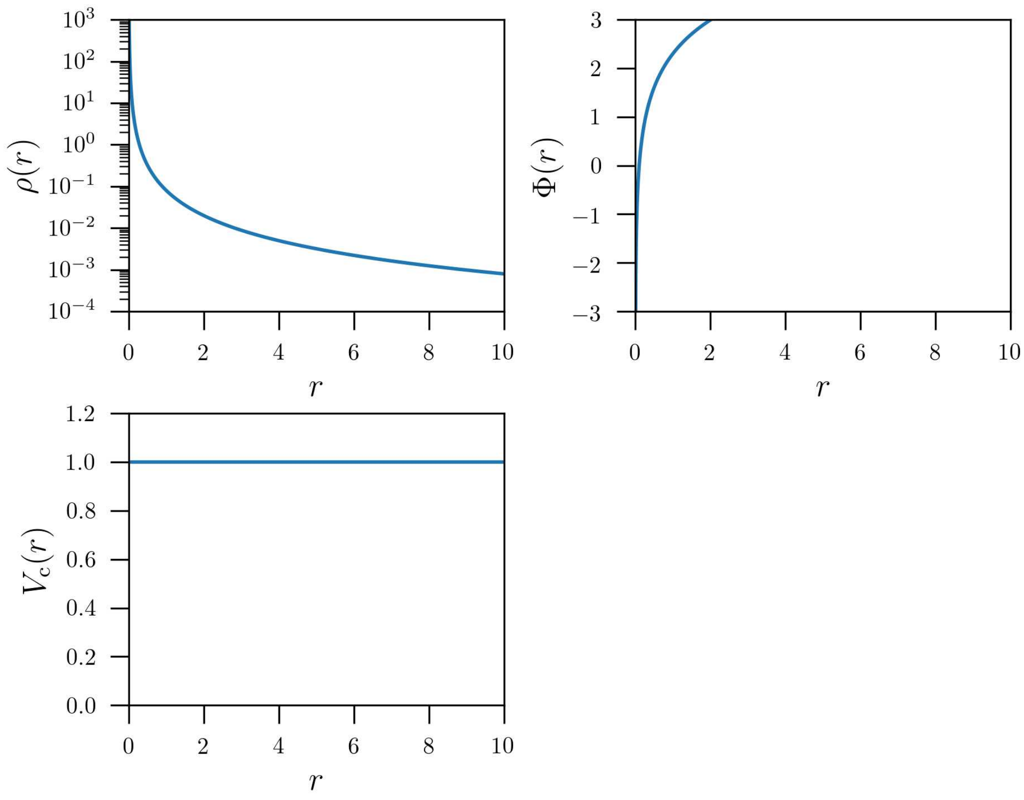
Isothermal sphere

$$\rho(r) = \rho_0 \frac{a^2}{r^2}$$

$$\Phi(r) = 4\pi G \rho_0 a^2 \ln\left(\frac{r}{a}\right)$$

$$M(r) = 4\pi \rho_0 a^2 \, r$$

$$V_c^2(r) = 4\pi G \rho_0 a^2$$



- often used for gravitational lens models
- But !
  - diverge towards the centre !
  - Infinite mass !

B Models with finite potential and density

$$\tilde{\rho} = \frac{\rho}{\rho_0} \qquad \tilde{r} = \frac{r}{r_0} \qquad r_0 = \sqrt{\frac{9\sigma^2}{4\pi G \rho_0}} \quad \text{(King radius)}$$

The Poisson equation becomes

$$\boxed{\frac{d}{d\tilde{r}}\left(\tilde{r}^2 \frac{d\ln\tilde{\rho}}{d\tilde{r}}\right) = -9\,\tilde{r}\,\tilde{\rho}}$$

Requires numerical integration

+ boundary conditions

$$\begin{cases} \bullet \ \tilde{\rho}(0) = 1 & \text{normalisation} \\ \bullet \ \left.\dfrac{d\tilde{\rho}}{d\tilde{r}}\right)_0 = 0 & \text{smooth} \end{cases}$$

Numerical solution of the non-singular isothermal sphere

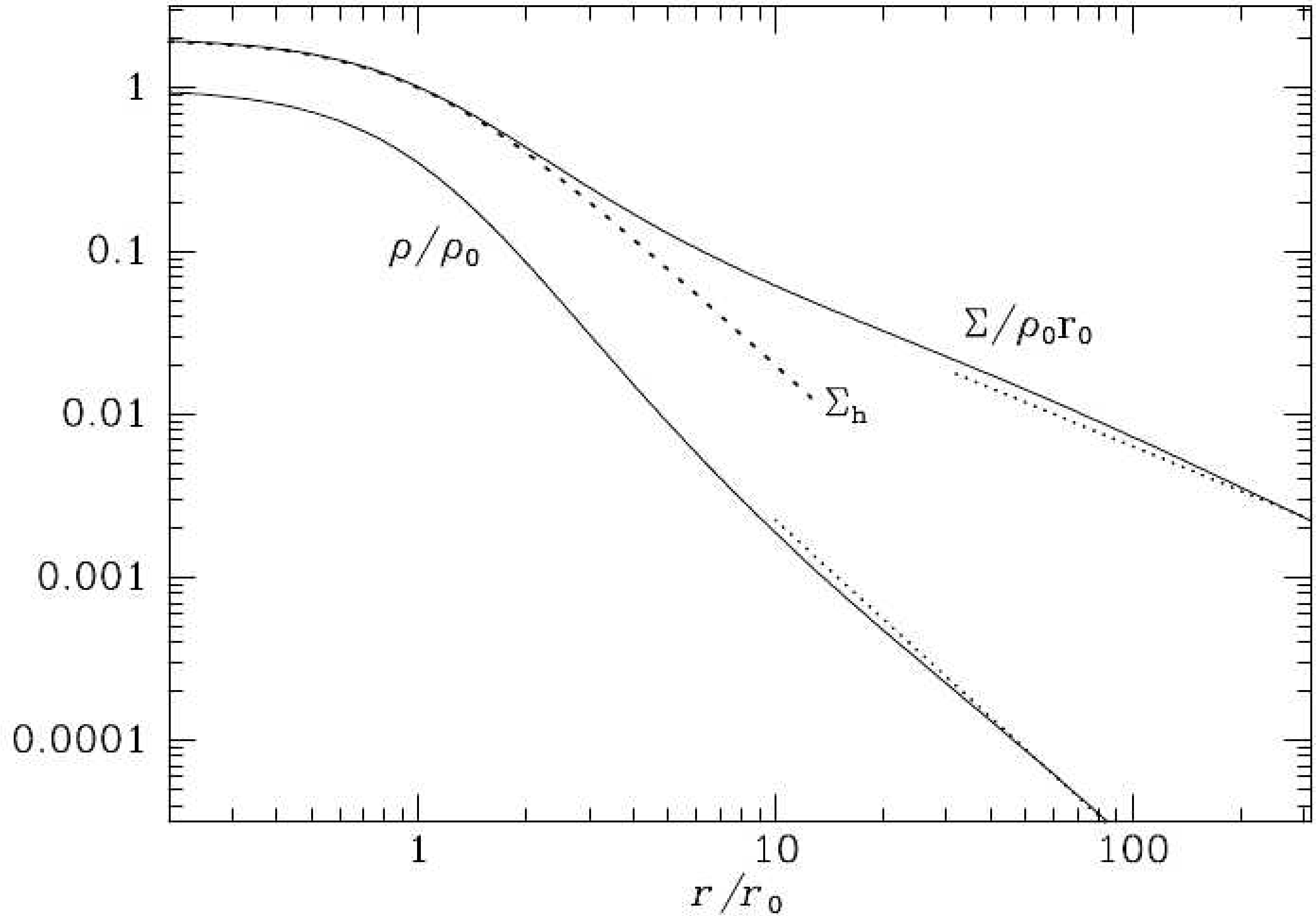


Figure 4.6 Volume (ρ/ρ_0) and projected ($\Sigma/\rho_0 r_0$) mass densities of the isothermal sphere. The dotted lines show the volume- and surface-density profiles of the singular isothermal sphere. The dashed curve shows the surface density of the modified Hubble model (4.109a).

Equilibria of collisionless systems

Anisotropic DFs in spherical systems

Spherical systems with anisothropic velocities

Ergodic DF : $f(\mathcal{E}) \Rightarrow \sigma_{ij} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$

If we know $\nu(r)$: Eddington's formula

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\,\pi^2} \frac{d}{d\mathcal{E}} \left[\int_0^{\mathcal{E}} \frac{d\Psi}{\sqrt{\mathcal{E}-\Psi}} \frac{d\nu}{d\Psi} \right]$$

or

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\,\pi^2} \left[\int_0^{\mathcal{E}} \frac{d\Psi}{\sqrt{\mathcal{E}-\Psi}} \frac{d^2\nu}{d\Psi^2} + \frac{1}{\sqrt{\mathcal{E}}} \left(\frac{d\nu}{d\Psi} \right)_{\Psi=0} \right]$$

Note : $f(\mathcal{E}) > 0$ only if $\int_0^{\mathcal{E}} \frac{d\Psi}{\sqrt{\mathcal{E}-\Psi}} \frac{d\nu}{d\Psi}$ is an increasing function of $\mathcal{E}$

⚠ for a given $\nu(r)$: no guarentee that $f(\mathcal{E}) > 0$ ⚠

By relaxing the assumption that $f = f(\varepsilon)$ (isothropic in v)

Ex: $f = f(\varepsilon, L = |\vec{L}|)$, we can ensure $f > 0$

Idea:

① Build a model based on circular orbits only.

By giving the appropriate weight to orbits at every radius, we can obtain a model with the desired $\nu(r)$

② Add it to an ergodic DF that generates $\nu(r)$

We can ensure that the sum of both DFs is positive.

Model based on circular orbits

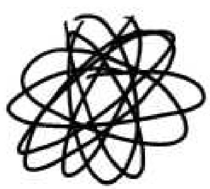
We split the model into a set of shells of radius r

- at each radius, we consider the corresponding circular orbits. For a given <u>density</u> and <u>potential</u>:

$$\begin{cases} - \text{ energy} & \mathcal{E}_{c,r} \\ - \text{ angular momentum} & L_c(\mathcal{E}_{c,r}) \end{cases}$$

- The DF of a spherical shell is thus:

$$f_{s,r}(\mathcal{E}, L) = \underbrace{\delta(\mathcal{E} - \mathcal{E}_{c,r})}_{\text{select the right energy}} \; \underbrace{\delta(L - L_c(\mathcal{E}_{c,r}))}_{\text{select the right ang. momentum}}$$



Note each shell contains orbit from all inclinaison (no selection on the direction)

Total DF Sum the contribution of all shells (integration over the radius) but as there is a bijective relation between r and $\varepsilon_{c,r}$ we can integrate over $\varepsilon_{c,r}$:

$$f_c(\varepsilon, L) = \int_0^{\varepsilon_{max}} d\varepsilon_{c,r}\, \delta(\varepsilon - \varepsilon_{c,r})\, \delta(L - L_c(\varepsilon_{c,r}))\, \underbrace{F(\varepsilon_{c,r})}_{\text{weight}}$$

$$\boxed{f_c(\varepsilon, L) = \delta(L - L_c(\varepsilon))\, F(\varepsilon)}$$

($= 0$ except when L corresponds to the angular momentum of the circular orbit of energy ε)

With a suitable weight $F(\varepsilon)$ $f_c(\varepsilon, L)$ generates $\nu(r)$

$$\nu(r) = \int d^3v\; F(\varepsilon)\, \delta\big(L - L_c(\varepsilon)\big) = 4\pi \int_0^{\infty} dv\, v^2\, F(\varepsilon)\, \delta\big(L - L_c(\varepsilon)\big)$$

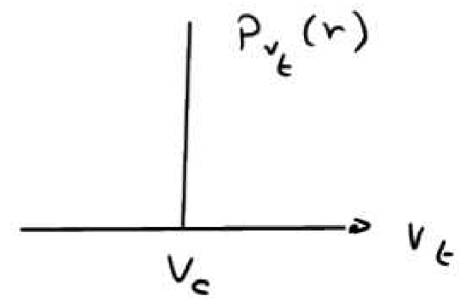
$$= 4\pi \int_{-\infty}^{\Psi} \sqrt{2(\Psi - \varepsilon)}\; F(\varepsilon)\, \delta\big(L - L_c(\varepsilon)\big)\, d\varepsilon = 4\pi \underbrace{\sqrt{2(\Psi - \varepsilon_{c,r})}}_{V_c(r)}\; F(\varepsilon_{c,r})$$

$$= 4\pi \sqrt{r \frac{\partial \Psi}{\partial r}}\; F\big(\varepsilon_{c,r}(r)\big)$$

$$\varepsilon = -\tfrac{1}{2} v^2 + \Psi$$

<u>Velocity dispersion</u> $\quad P_v(\varepsilon) = \dfrac{1}{4\pi V_c}\, \delta\big(L - L_c(\varepsilon)\big)$

- All orbits are purely tangential (circular)



- $V_r = 0$
- $\sigma_r = 0$

Idea: If $f_i(\varepsilon)$ is an ergodic DF

we can define new DFs : (Note: we ensure $\nu(r) = \int f_\alpha \, d^3v$)

$$\boxed{f_\alpha(\varepsilon, L) = \alpha f_i(\varepsilon) + (1-\alpha) f_c(\varepsilon, L)}$$

$\alpha = 0$: circular orbits $\sigma_\theta = \sigma_\varphi \neq 0 \; , \; \sigma_r = 0$

eccentricity of orbits increases

$\alpha = 1$: ergodic (isothrope) $\sigma_\theta = \sigma_\varphi = \sigma_r$

$\alpha > 1$: more elongated orbits "radial" $\sigma_\theta = \sigma_\varphi < \sigma_r$

⚠ as long as $f_\alpha > 0$

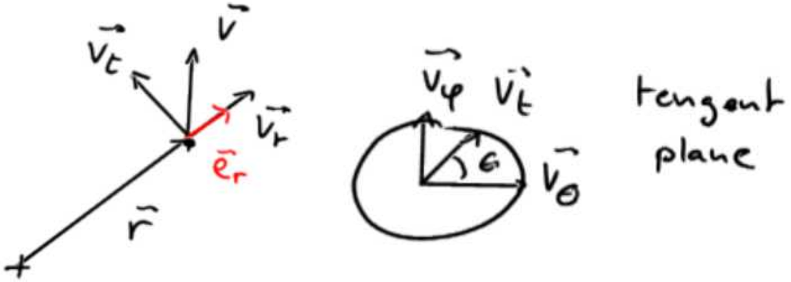
If $f_i(\varepsilon) < 0$ we can then ensure $f_\alpha(\varepsilon, L) > 0$ as

1) $f_c(\varepsilon, L) > 0$

2) $(1-\alpha) > 0$ $\alpha \in [0, 1]$

i.e giving more weight to circular orbits

Definition: anisotropy parameter



$$\beta := 1 - \frac{\sigma_\theta^2 + \sigma_\phi^2}{2\sigma_r^2} = 1 - \frac{\sigma_t^2}{2\sigma_r^2}$$

$\beta = -\infty$

- Circular orbits

  $\sigma_\theta = \sigma_\phi \neq 0, \sigma_r = 0$

$\beta = 0$

- Isotrope ergodic

  $\sigma_\theta = \sigma_\phi = \sigma_r = \frac{1}{\sqrt{2}}\sigma_t$

$\beta = 1$

- Radial orbits

  $\sigma_\theta = \sigma_\phi = 0, \sigma_r \neq 0$

- tangentially biased orbits
  $\sigma_\theta = \sigma_\phi > \sigma_r$

- radially biased orbits
  $\sigma_\theta = \sigma_\phi < \sigma_r$

Models with constant anisotropy

$$\boxed{f(\mathcal{E}, L) = f_1(\mathcal{E})\, L^{\gamma} = f_1(\mathcal{E})\, L^{-2\beta}} \qquad f_1(\mathcal{E}) > 0$$

Can we find an expression for $f_1(\mathcal{E})$, for a given $\Phi(r)$ and $\rho(r)$?

From $\quad \nu(r) = \int d^3\vec{v}\, f_1(\mathcal{E})\, L^{-2\beta}$

$$\boxed{\frac{2^{\beta - 1/2}}{2\pi I_\beta}\, r^{2\beta}\, \nu(\Psi) = \int_0^{\Psi} d\mathcal{E}\, \frac{f_1(\mathcal{E})}{(\Psi - \mathcal{E})^{\beta - 1/2}}}$$

<u>Note:</u> Differenciating with respect to Ψ, we can obtain an Abel equation and the equivalent of the Eddington formula.

Density : $\nu(r) = \int d^3\vec{v}\; f_-(\varepsilon)\, L^{-2\beta}$

integration using polar coord. in velocity space :

$$\left\{ \begin{array}{l} v_r = v \cos\eta \\ v_\theta = v \sin\eta \cos\varphi \\ v_\varphi = v \sin\eta \sin\varphi \end{array} \right. \qquad \begin{array}{l} L = r\sqrt{v_\theta^2 + v_\varphi^2} = r v \sin\eta \\ d^3\vec{v} = dv_r\, dv_\theta\, dv_\varphi\; v^2 \sin\eta \end{array}$$

$$\nu(r) = \int d^3\vec{v}\; f_-(\varepsilon)\, L^{-2\beta}$$

$$= 2\pi \int_0^\pi d\eta \sin\eta \int_0^\infty dv\, v^2\, f_-\left(\psi(r) - \tfrac{1}{2}v^2\right) L^{-2\beta}$$

$$= \frac{2\pi}{r^{2\beta}} \underbrace{\int_0^\pi d\eta \sin^{1-2\beta}\eta}_{\sqrt{\pi}\,\frac{(-\beta)!}{(\frac{1}{2}-\beta)!} \;:=\; I_\beta} \int_0^\infty dv\, v^{2-2\beta} f_-\left(\psi(r) - \tfrac{1}{2}v^2\right)$$

$(\,!\ \beta < 1)$

And integrating through the energy $\varepsilon = \psi - \frac{1}{2} v^2$

$$\begin{cases} v = \sqrt{2(\psi - \varepsilon)} \qquad dv = \dfrac{-1}{\sqrt{2(\psi-\varepsilon)}}\, d\varepsilon \\ \frac{1}{2} v^2 + \phi = \phi_0 - \varepsilon \end{cases}$$

+ $\nu(r)$ is a monotonic function of ψ

$$\boxed{\frac{2^{\beta - 1/2}}{2\pi I_\beta}\, r^{2\beta}\, \nu(\psi) = \int_0^\psi d\varepsilon \, \frac{\rho_1(\varepsilon)}{(\psi - \varepsilon)^{\beta - 1/2}}}$$

#

$$\boxed{\text{Case} \quad \beta = \frac{1}{2}}$$

$\sigma_\theta^2 = \sigma_\phi^2 = \frac{1}{2}\sigma_r^2$ (radially biased)

$$\boxed{\frac{2^{\beta-1/2}}{2\pi I_\beta} r^{2\beta} \nu(\Psi) = \int_0^\Psi d\mathcal{E} \frac{f_1(\mathcal{E})}{(\Psi-\mathcal{E})^{\beta-1/2}}}$$

becomes

$$\frac{1}{2\pi^2} r \nu(\Psi) = \int_0^\Psi d\mathcal{E} \, f_1(\mathcal{E})$$

and $\frac{d}{d\Psi}$ gives :

$$\boxed{f_1(\Psi) = \frac{1}{2\pi^2} \frac{d}{d\Psi}(r\nu)}$$

$$\boxed{\text{Case} \quad \beta = -\frac{1}{2}}$$

$$\sigma_\theta^2 = \sigma_\phi^2 = \frac{3}{2}\sigma_r^2 \quad (\text{tangentially biased})$$

$$\boxed{\frac{2^{\beta-1/2}}{2\pi I_\beta} r^{2\beta} \nu(\psi) = \int_0^\psi d\varepsilon \, \frac{f_1(\varepsilon)}{(\psi-\varepsilon)^{\beta-1/2}}}$$

becomes

$$\frac{1}{2\pi^2} \frac{\nu(\psi)}{r} = \int_0^\psi d\varepsilon \, f_1(\varepsilon)\,(\psi-\varepsilon)$$

and $\frac{d}{d\psi}$ gives :

$$\boxed{f_1(\psi) = \frac{1}{2\pi^2} \frac{d}{d\psi^2}\left(\nu/r\right)}$$

Application to the Hernquist model

$$\frac{r}{a} = \frac{1}{\tilde{\psi}} - 1 \qquad \text{where } \tilde{\psi}(r) = \frac{\psi(r)}{GM}\,a$$

- $\beta = \frac{1}{2}$

$$\boxed{f_1(\varepsilon) = \frac{3\,\tilde{\varepsilon}^2}{4\pi^3 GMa}} \qquad \text{with } \tilde{\varepsilon} = \frac{\varepsilon a}{GM}$$

- $\beta = -\frac{1}{2}$

$$\boxed{f_1(\varepsilon) = \frac{1}{4\pi^3 (GMa)^2}\,\frac{d^2}{d\hat{\varepsilon}^2}\left(\frac{\hat{\varepsilon}^5}{(1-\hat{\varepsilon})^2}\right)}$$

Line of sight velocity of Hernquist models with three different anisotropies (β)

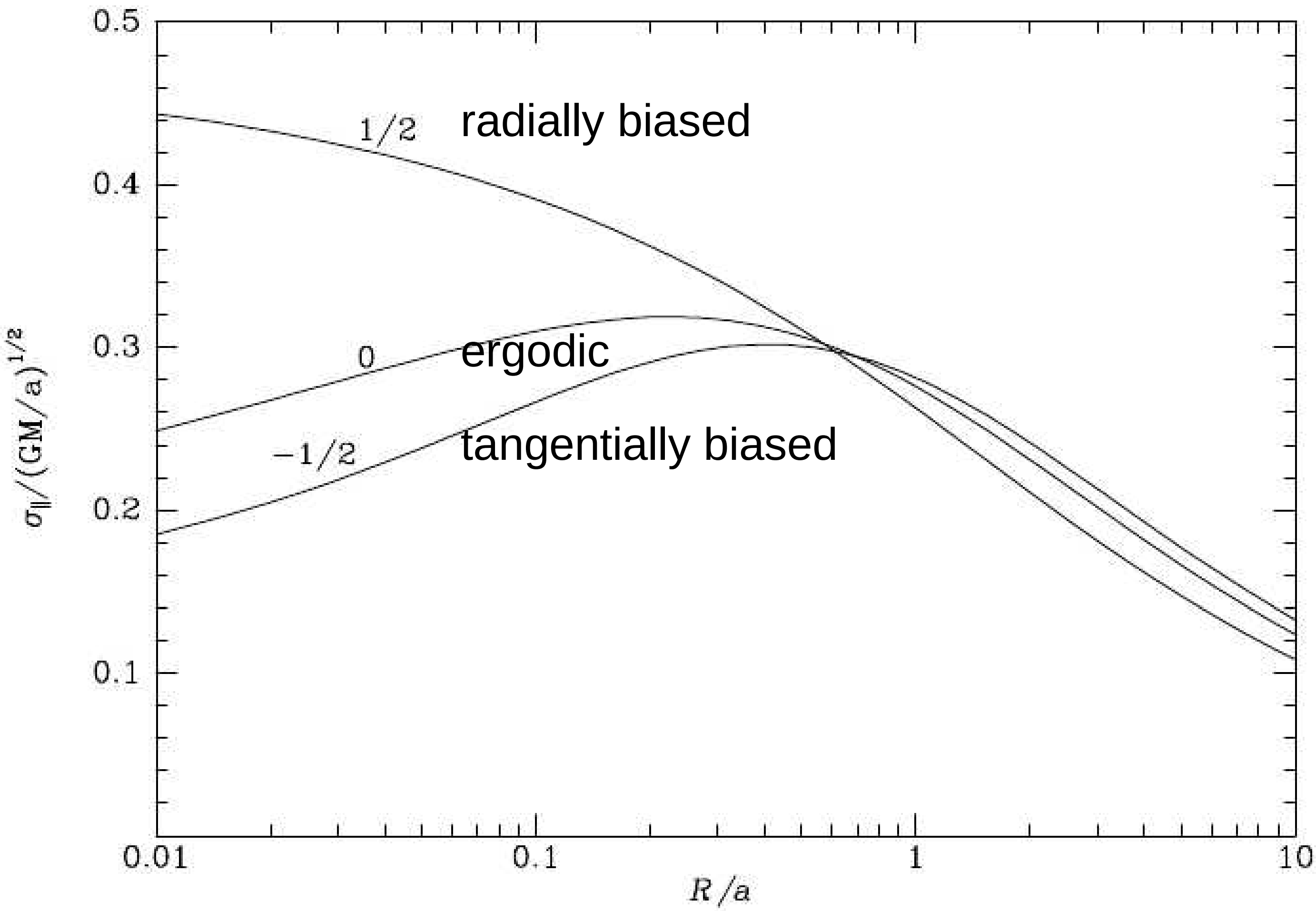
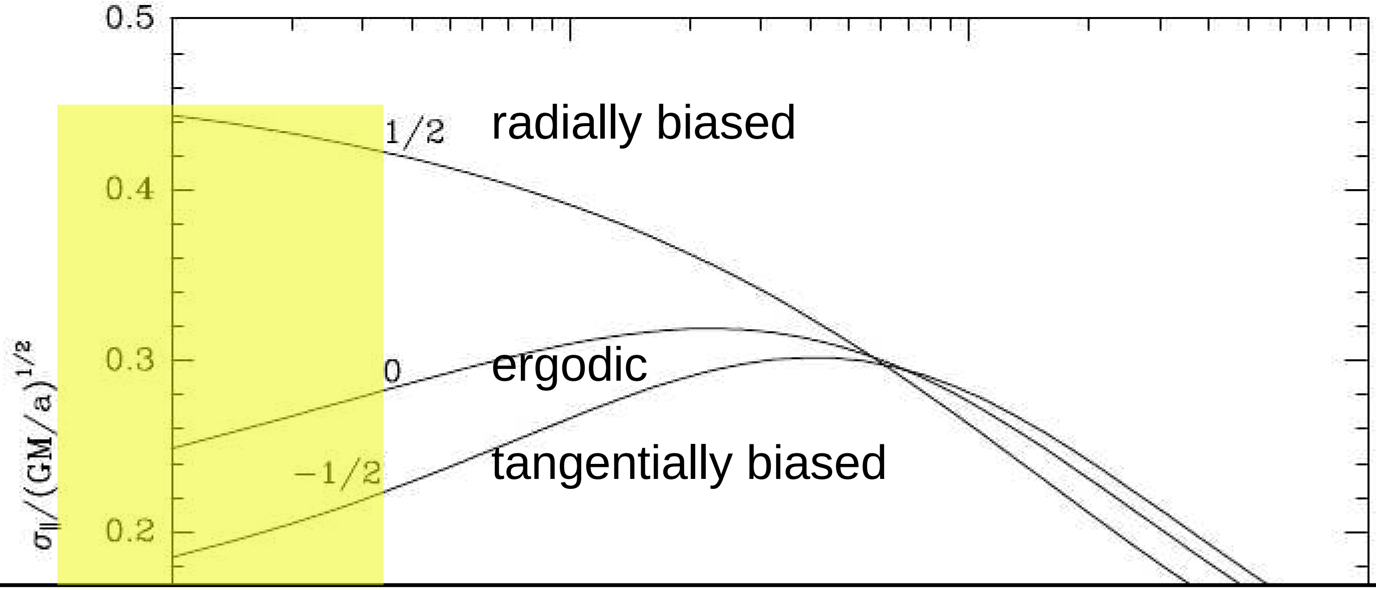
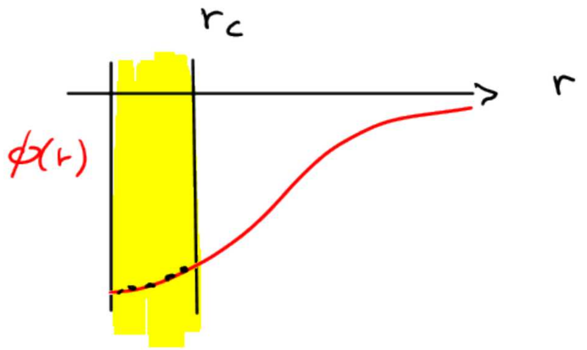


Figure 4.4 Line-of-sight velocity dispersion as a function of projected radius, from spatially identical systems that have different DFs. In each system the density and potential are those of the Hernquist model and the anisotropy parameter β of equation (4.61) is independent of radius. The curves are labeled by the relevant value of β. In the isotropic system, the velocity dispersion falls as one approaches the center (cf. Problem 4.14).

Line of sight velocity of Hernquist models with three different anisotropies (β)

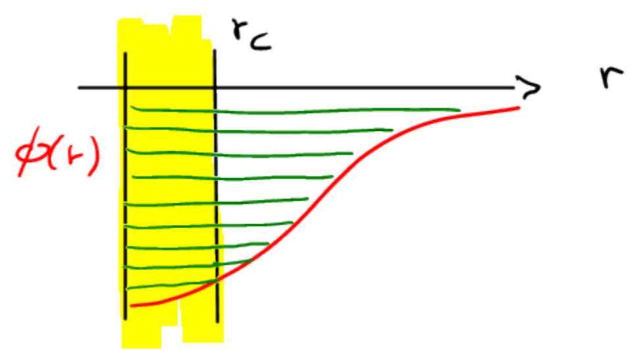


<u>at the center</u>: circular orbits, are only low energy orbits



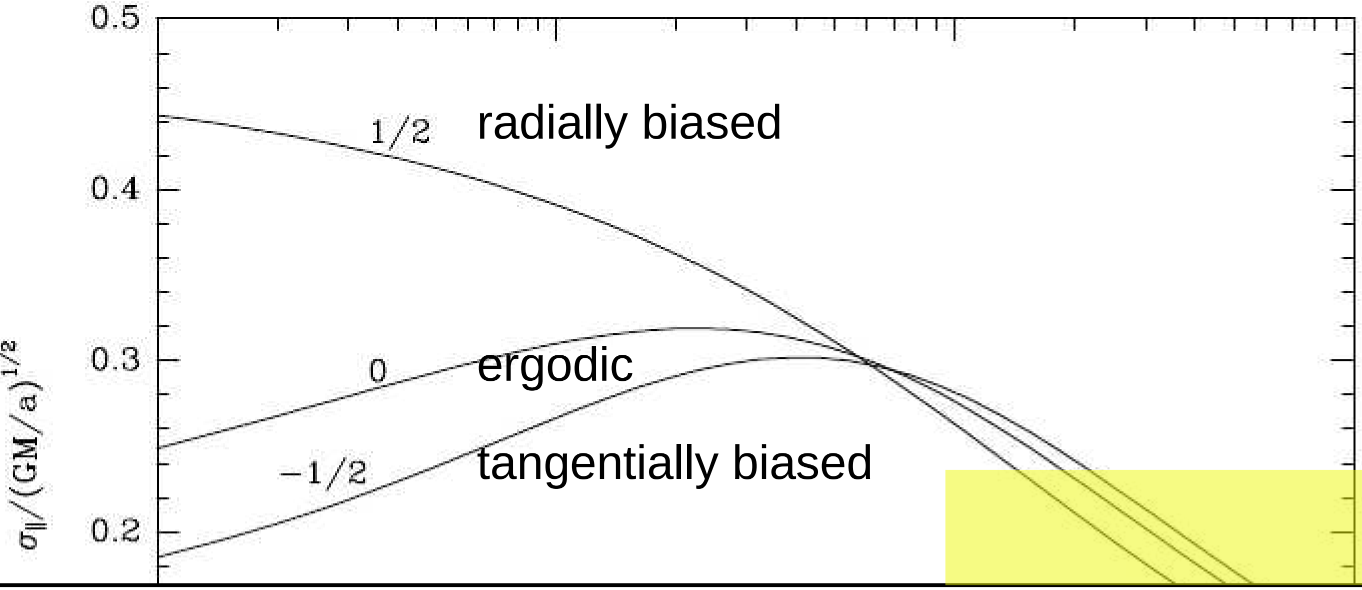
→ low range in velocities
→ low velocity dispersion

<u>at the center</u>: radial orbits, span all energies, including high ones

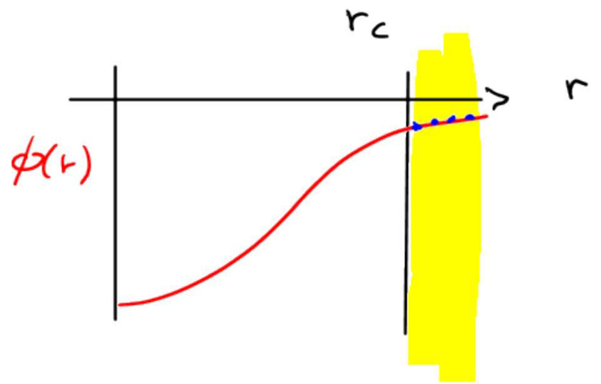


→ large range in velocities
→ high velocity dispersion

Line of sight velocity of Hernquist models with three different anisotropies (β)

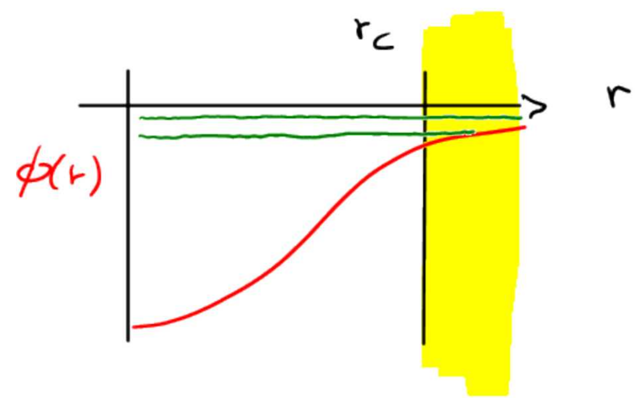


in the periphery: circular orbits, are only high energy orbits



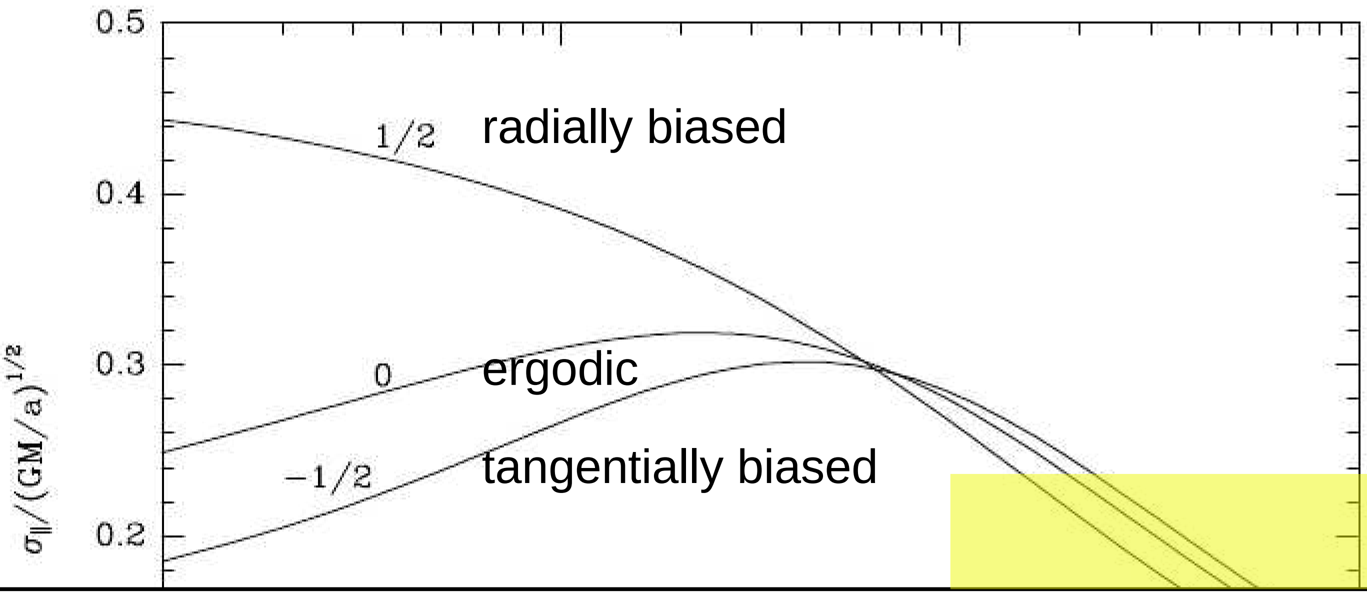
→ low range in velocities
→ low velocity dispersion

in the periphery: radial orbits, span a small energy range



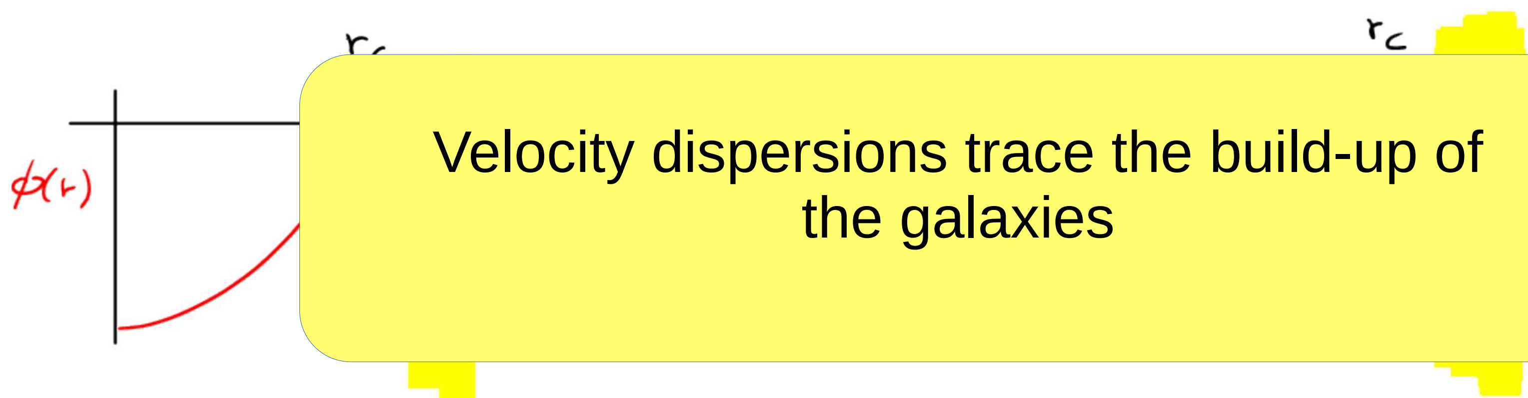
→ low range in velocities
→ low velocity dispersion

Line of sight velocity of Hernquist models with three different anisotropies (β)



in the periphery: circular orbits, are only low energy orbits

in the periphery: radial orbits, span a small Energy range



Velocity dispersions trace the build-up of the galaxies

$\rightarrow$ low range in velocities
$\rightarrow$ low velocity dispersion

$\rightarrow$ low range in velocities
$\rightarrow$ low velocity dispersion

Equilibria of collisionless systems

Jeans Equations

The Jeans Equations

- From observations, we usually obtain velocity moments:

  Examples : mean velocity $\bar{v}_i$

  velocity dispersions $\overline{v_i v_j} \equiv \sigma_{ij}$

- Computing moments from a DF is "easy":

$$\bar{v}_i = \frac{1}{\nu(\vec{x})} \int v_i \, f(\vec{x}, \vec{v}) \, d^3\vec{v}$$

- Obtaining a DF compatible with an observed $\nu(\vec{x})$ ($\rho(\vec{x})$) is less easy and solutions are often not unique.

Our goal — Find a method that let infer moments from stellar systems, without recovering the DF.

Idea — Compute moments of the collisionless Boltzman equation.

In carthesian coordinates

$$\frac{\partial f}{\partial t} + \vec{v}\,\frac{\partial f}{\partial \vec{x}} - \vec{\nabla}\phi\,\frac{\partial f}{\partial \vec{v}} = 0$$

$$\boxed{\frac{\partial}{\partial t} f + \sum_i v_i \frac{\partial f}{\partial x_i} - \sum_i \frac{\partial \phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0}$$

Zeroth moment

$$\frac{\partial}{\partial t} f + \sum_i v_i \frac{\partial f}{\partial x_i} - \sum_i \frac{\partial \phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0$$

integrate over velocities

$$\int \frac{\partial}{\partial t} f \, d^3v + \sum_i \int d^3v \; v_i \frac{\partial f}{\partial x_i} - \sum_i \frac{\partial \phi}{\partial x_i} \int d^3v \frac{\partial f}{\partial v_i} = 0$$

$$\frac{\partial}{\partial t} \underbrace{\int f \, d^3v}_{\nu(\vec{x})} + \sum_i \frac{\partial}{\partial x_i} \int d^3v \; v_i f - \sum_i \frac{\partial \phi}{\partial x_i} \underbrace{\oint_S dS_i f}_{=0} = 0$$

v_i does not dep. on x_i (canonical coords)

div. theorem ⊛ + $f(x, v, t) = 0$ for $v \to \infty$ $= 0$

We get

$$\boxed{\frac{\partial}{\partial t} \nu(\vec{x}) + \sum_i \frac{\partial}{\partial x_i} \left(\nu \bar{v}_i \right) = 0}$$

continuity equation for $\nu(x)$

$$\boxed{\frac{\partial}{\partial t} \nu + \vec{\nabla} \cdot \left(\nu \vec{\bar{v}} \right) = 0}$$

$$\left(\frac{\partial}{\partial t} \rho + \vec{\nabla} \cdot \left(\rho \vec{v} \right) \right) \qquad \nu \equiv \rho, \quad \vec{\bar{v}} \equiv \vec{v}$$

⊛ div. theorem $\int d^3x \, \vec{\nabla} \vec{F} = \int d^2\vec{S} \cdot \vec{F}$

for $\vec{F} = f \vec{e}_j$ $\quad \int d^3x \frac{\partial f}{\partial x_j} = \int d^2 S_j \, f$

$$\boxed{\frac{\partial}{\partial t} f + \sum_i v_i \frac{\partial f}{\partial x_i} - \sum_i \frac{\partial \phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0}$$

First moment

multiply by v_j and integrate over velocities

$$\frac{\partial}{\partial t} \underbrace{\int d^3v \; v_j \; f}_{\nu \bar{v}_j} + \underbrace{\int d^3v \sum_i v_i v_j \frac{\partial f}{\partial x_i}}_{1} - \sum_i \frac{\partial \phi}{\partial x_i} \underbrace{\int d^3v \; v_i \frac{\partial f}{\partial v_i}}_{2 \; = \; \frac{\partial \phi}{\partial x_j}\nu} = 0$$

1 $\quad \int d^3v \sum_i v_i v_j \frac{\partial f}{\partial x_i} = \sum_i \frac{\partial}{\partial x_i} \int d^3v \; v_i v_j f = \sum_i \frac{\partial}{\partial x_i} \left(\overline{v_i v_j} \; \nu \right)$

2 $\quad \int d^3v \frac{\partial}{\partial v_i}\left(v_j f \right) = \underbrace{\int d^3v \; v_i \frac{\partial f}{\partial v_i}}_{2} + \underbrace{\int d^3v \; f \underbrace{\frac{\partial v_j}{\partial v_i}}_{\delta_{ij}}}_{\delta_{ij}\nu}$

$\oint dS_i \; v_j f = 0$

$$\boxed{\frac{\partial}{\partial t}\left(\bar{v}_j \nu \right) + \sum_i \frac{\partial}{\partial x_i}\left(\overline{v_i v_j} \; \nu \right) + \nu \frac{\partial \phi}{\partial x_j} = 0}$$

Using the continuity equation multiplied by $\bar{v}_j$

$$\bar{v}_j \left(\frac{\partial}{\partial t} \nu(\vec{x}) + \sum_i \frac{\partial}{\partial x_i} (\nu \bar{v}_i) \right) = 0$$

and substracting it from the previous result

$$\underbrace{\frac{\partial}{\partial t}(\bar{v}_j \nu) - \bar{v}_j \frac{\partial}{\partial t} \nu}_{\nu \frac{\partial}{\partial t}(\bar{v}_j)} + \underbrace{\sum_i \frac{\partial}{\partial x_i} (\overline{v_i v_j}\, \nu) - \bar{v}_j \sum_i \frac{\partial}{\partial x_i} (\nu \bar{v}_i)}_{1} + \nu \frac{\partial \phi}{\partial x_j} = 0$$

with $\sigma_{ij}^2 = \overline{v_i v_j} - \bar{v}_i \bar{v}_j$

$$1 = \sum_i \frac{\partial}{\partial x_i} (\sigma_{ij}^2 \nu) + \underbrace{\sum_i \frac{\partial}{\partial x_i} (\bar{v}_i \bar{v}_j \nu)} - \bar{v}_j \sum_i \frac{\partial}{\partial x_i} (\nu \bar{v}_i)$$

$$\nu \sum_i \bar{v}_i \frac{\partial}{\partial x_i} \bar{v}_j + \underbrace{\sum_i \bar{v}_j \frac{\partial}{\partial x_i}(\nu \bar{v}_i) - \bar{v}_j \sum_i \frac{\partial}{\partial x_i} (\nu \bar{v}_i)}_{=0}$$

$$\nu \frac{\partial}{\partial t}(\bar{v}_j) + \nu \sum_i \bar{v}_i \frac{\partial}{\partial x_i} \bar{v}_j = - \sum_i \frac{\partial}{\partial x_i}(\sigma_{ij}^2 \nu) - \nu \frac{\partial \phi}{\partial x_j}$$

Jeans 1919

Interpretation — Euler equation in hydrodynamics

Lagrangian form

$$\frac{d}{dt} \vec{v} = - \frac{\vec{\nabla} P}{\rho} - \vec{\nabla} \phi$$

Eulerian form

$$(*) \quad \frac{\partial}{\partial t} \vec{v} + \vec{v} \cdot \vec{\nabla} \vec{v} = - \frac{\vec{\nabla} P}{\rho} - \vec{\nabla} \phi$$

$$\rho \frac{\partial}{\partial t} \vec{v} + \rho \vec{v} \cdot \vec{\nabla} \vec{v} = - \vec{\nabla} P - \rho \vec{\nabla} \cdot \phi$$

"j" component only

$$\boxed{\rho \frac{\partial}{\partial t} v_j + \rho \sum_i v_i \frac{\partial v_j}{\partial x_i} = - \frac{\partial P}{\partial x_j} - \rho \frac{\partial \phi}{\partial x_j}}$$

$$(*) \quad \frac{d v_i(x, y, z)}{dt} = \frac{\partial v_i}{\partial t} + \sum \frac{\partial v_i}{\partial x} \dot{x}$$

Both equations are similar

if $\boxed{\rho = \nu}$ $\boxed{V_i = \bar{V}_i}$ $\boxed{\frac{\partial p}{\partial x_j} = \sum_i \frac{\partial}{\partial x_i}(\sigma_{ij}^2 \nu)}$

$$\begin{pmatrix} p & & \\ & p & \\ & & p \end{pmatrix} \equiv \begin{pmatrix} \sigma_{xx}^2 & \sigma_{xy}^2 & \sigma_{xz}^2 \\ \sigma_{yx}^2 & \sigma_{yy}^2 & \sigma_{yz}^2 \\ \sigma_{zx}^2 & \sigma_{zy}^2 & \sigma_{zz}^2 \end{pmatrix} \nu$$

anisotropic stress tensor

(symmetric)

diagonal in an appropriate rest frame $\begin{pmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \sigma_3^2 \end{pmatrix} \nu$

Thus $\boxed{\frac{\partial p}{\partial x_j} = \frac{\partial}{\partial x_j}(\sigma_{jj}^2 \nu)}$

Note: it is possible to show that for an ergodic system,

$$p = \int_0^\rho d\rho' \rho' \frac{\partial f}{\partial \rho'}$$

leads to

$$p = \sigma^2 \nu$$

Comments

$f(\bar{x},\bar{v})$ is unknown

2 known quantities : $\rho(\bar{x})$, $\phi(\bar{x})$

6 unknown quantities : $\bar{v}_x$ $\bar{v}_y$ $\bar{v}_z$, σ_{xx} , σ_{yy} , σ_{zz} (assuming it is diagonal)

4 equations : zeroth moment (1) + first moment (3)

$\boxed{\text{The Jeans equations are not closed !}}$

- if we multiply the CB by $v_i v_j$ → new terms $\overline{v_i v_j v_k}$
  → not a solution
- we need to do some assumptions (closure conditions)

example : σ_{ij} (3) → σ (1) ok if f is ergodic

Equilibria of collisionless systems

“Static” Jeans Equations for spherical systems

Canonical momenta

$$\begin{cases} p_r = \dot{r} = v_r \\ p_\theta = r^2 \dot{\theta} = r v_\theta \\ p_\phi = r^2 \sin^2(\theta \dot{\phi}) = r \sin(\theta)\, v_\phi \end{cases}$$

The static Collisionless Boltzmann Equation, for spherical systems

$$\cancel{\frac{\partial f}{\partial t}} + p_r \frac{\partial f}{\partial r} + \frac{p_\theta}{r^2} \frac{\partial f}{\partial \theta} + \cancel{\frac{p_\phi}{r^2 \sin^2(\theta)} \frac{\partial f}{\partial \phi}} - \left(\frac{\partial \Phi}{\partial r} - \frac{p_\theta^2}{r^3} - \frac{p_\phi^2}{r^3 \sin^2(\theta)} \right) \frac{\partial f}{\partial p_r} - \left(\cancel{\frac{\partial \Phi}{\partial \theta}} - \frac{p_\phi^2 \cos(\theta)}{r^2 \sin^3(\theta)} \right) \frac{\partial f}{\partial p_\theta} - \cancel{\frac{\partial \Phi}{\partial \phi}} \frac{\partial f}{\partial p_\phi} = 0$$

f can depend on θ as $p_\phi = r \sin(\theta)\, v_\phi$

Zeroth order moment of the Jeans Equation

$$\frac{\partial}{\partial r} \left(\sin(\theta) \nu \overline{v_r} \right) = \frac{\partial}{\partial \theta} \left(\sin(\theta) \nu \overline{v_\theta} \right)$$

The Jeans equations for spherical systems

Canonical momenta

$$\begin{cases} p_r = \dot{r} = v_r \\ p_\theta = r^2\dot{\theta} = r v_\theta \\ p_\phi = r^2 \sin^2(\theta\dot{\phi}) = r\sin(\theta)\, v_\phi \end{cases}$$

The static Collisionless Boltzmann Equation, for spherical systems

$$\cancel{\frac{\partial f}{\partial t}} + p_r\frac{\partial f}{\partial r} + \frac{p_\theta}{r^2}\frac{\partial f}{\partial \theta} + \cancel{\frac{p_\phi}{r^2\sin^2(\theta)}}\frac{\partial f}{\partial \phi} - \left(\frac{\partial \Phi}{\partial r} - \frac{p_\theta^2}{r^3} - \frac{p_\phi^2}{r^3\sin^2(\theta)}\right)\frac{\partial f}{\partial p_r} - \left(\cancel{\frac{\partial \Phi}{\partial \theta}} - \frac{p_\phi^2\cos(\theta)}{r^2\sin^3(\theta)}\right)\frac{\partial f}{\partial p_\theta} - \cancel{\frac{\partial \Phi}{\partial \phi}}\frac{\partial f}{\partial p_\phi} = 0$$

f can depend on θ as $p_\phi = r\sin(\theta)\, v_\phi$

Zeroth order moment of the Jeans Equation

$$\frac{\partial}{\partial r}\left(\sin(\theta)\nu\overline{v_r}\right) = \frac{\partial}{\partial \theta}\left(\sin(\theta)\nu\overline{v_\theta}\right)$$

if $f = f(H)$ or $f(H, L) \Rightarrow \overline{v_r} = \overline{v_z} = \overline{v_\theta} = 0$

$$\overline{v_r^2} = \sigma_r^2 \quad \overline{v_\theta^2} = \sigma_\theta^2 \quad \overline{v_\phi^2} = \sigma_\phi^2$$

The Jeans equations for spherical systems

Canonical momenta

$$\begin{cases} p_r = \dot{r} = v_r \\ p_\theta = r^2\dot{\theta} = r v_\theta \\ p_\phi = r^2 \sin^2(\theta\dot{\phi}) = r\sin(\theta)\, v_\phi \end{cases}$$

The static Collisionless Boltzmann Equation, for spherical systems

$$\cancel{\frac{\partial f}{\partial t}} + p_r \frac{\partial f}{\partial r} + \frac{p_\theta}{r^2}\frac{\partial f}{\partial \theta} + \cancel{\frac{p_\phi}{r^2 \sin^2(\theta)}\frac{\partial f}{\partial \phi}} - \left(\frac{\partial \Phi}{\partial r} - \frac{p_\theta^2}{r^3} - \frac{p_\phi^2}{r^3 \sin^2(\theta)}\right)\frac{\partial f}{\partial p_r} - \left(\cancel{\frac{\partial \Phi}{\partial \theta}} - \frac{p_\phi^2 \cos(\theta)}{r^2 \sin^3(\theta)}\right)\frac{\partial f}{\partial p_\theta} - \cancel{\frac{\partial \Phi}{\partial \phi}\frac{\partial f}{\partial p_\phi}} = 0$$

f can depend on θ as $p_\phi = r\sin(\theta)\, v_\phi$

Zeroth order moment of the Jeans Equation

EXERCICE

$$0 = 0$$

if $f = f(H)$ or $f(H, L) \Rightarrow \overline{v_r} = \overline{v_z} = \overline{v_\theta} = 0$

$$\overline{v_r^2} = \sigma_r^2 \quad \overline{v_\theta^2} = \sigma_\theta^2 \quad \overline{v_\phi^2} = \sigma_\phi^2$$

First order moment of the Jeans Equation

EXERCICE

$$\frac{\partial}{\partial r}\left(\nu \overline{v_r^2}\right) + \nu\left(\frac{\partial \Phi}{\partial r} + \frac{2\overline{v_r^2} - \overline{v_\theta^2} - \overline{v_\phi^2}}{r}\right) = 0$$

or

$$\boxed{\frac{\partial}{\partial r}\left(\nu \overline{v_r^2}\right) + 2\frac{\beta}{r}\nu\overline{v_r^2} = -\nu\frac{\partial \Phi}{\partial r}}$$

where

$$\beta = 1 - \frac{\overline{v_\theta^2} + \overline{v_\phi^2}}{2\overline{v_r^2}} = 1 - \frac{\overline{v_t^2}}{2\overline{v_r^2}}$$

Discussion

$$\boxed{\frac{\partial}{\partial r}\left(\nu\,\sigma_r^2\right) + \nu\left(\frac{\partial \phi}{\partial r} + \frac{2\sigma_r^2 - \sigma_\theta^2 - \sigma_\varphi^2}{r}\right) = 0}$$

Case $\boxed{\sigma_r = \sigma_\varphi = \sigma_\theta}$ Ergodic

$$\Rightarrow \frac{1}{\nu}\frac{\partial}{\partial r}\left(\nu\,\sigma_r^2\right) = -\frac{\partial \phi}{\partial r}$$

$$\equiv \frac{\tilde{\nabla} P}{\rho} = \vec{F}_{grav}$$

Note : for σ = cte , we should recover the isothermal sphere

$$\boxed{\frac{\partial}{\partial r}\left(\nu\,\sigma_r^2\right) + \nu\left(\frac{\partial\phi}{\partial r} + \frac{2\sigma_r^2 - \sigma_\theta^2 - \sigma_\varphi^2}{r}\right) = 0}$$

Discussion

Case $\boxed{\sigma_r = 0}$ $\rightarrow$ $\sigma_t^2 = r\,\frac{\partial\phi}{\partial r}$

interpretation

only circular orbits

$v_t^2 = r\,\frac{\partial\phi}{\partial r}$

but from all possible planes

Demonstration

associated dispersion : in the tangential plane

$$v_\varphi = v_t \cos\eta \qquad \sigma_\varphi^2 = \frac{1}{2\pi}\int v_t^2 \cos^2\eta \; d\eta = \frac{1}{2} v_t^2$$

$$v_\theta = v_t \sin\eta \qquad \sigma_\theta^2 = \qquad\qquad = \frac{1}{2} v_t^2$$

thus $\sigma_t^2 := \sigma_\varphi^2 + \sigma_\theta^2 = v_t^2$

#

Discussion

$$\boxed{\frac{\partial}{\partial r}\left(\nu\,\sigma_r^{2}\right) + \nu\left(\frac{\partial \phi}{\partial r} + \frac{2\sigma_r^{2} - \sigma_\theta^{2} - \sigma_\varphi^{2}}{r}\right) = 0}$$

Case

$\boxed{\sigma_t = 0}$ $\quad\Rightarrow\quad \dfrac{1}{\nu}\dfrac{\partial}{\partial r}\left(\nu\,\sigma_r^{2}\right) + \dfrac{2\sigma_r^{2}}{r} = -\dfrac{\partial \phi}{\partial r}$

purely radial orbits

The Jeans equations for spherical systems

$$\boxed{\frac{\partial}{\partial r}\left(\nu\sigma_r^2\right) + 2\frac{\beta}{r}\nu\sigma_r^2 = -\nu\frac{\partial\Phi}{\partial r}}$$

$$\boxed{r^{-2\beta}\frac{\partial}{\partial r}\left(\nu\sigma_r^2 r^{2\beta}\right) = -\nu\frac{\partial\Phi}{\partial r}}$$

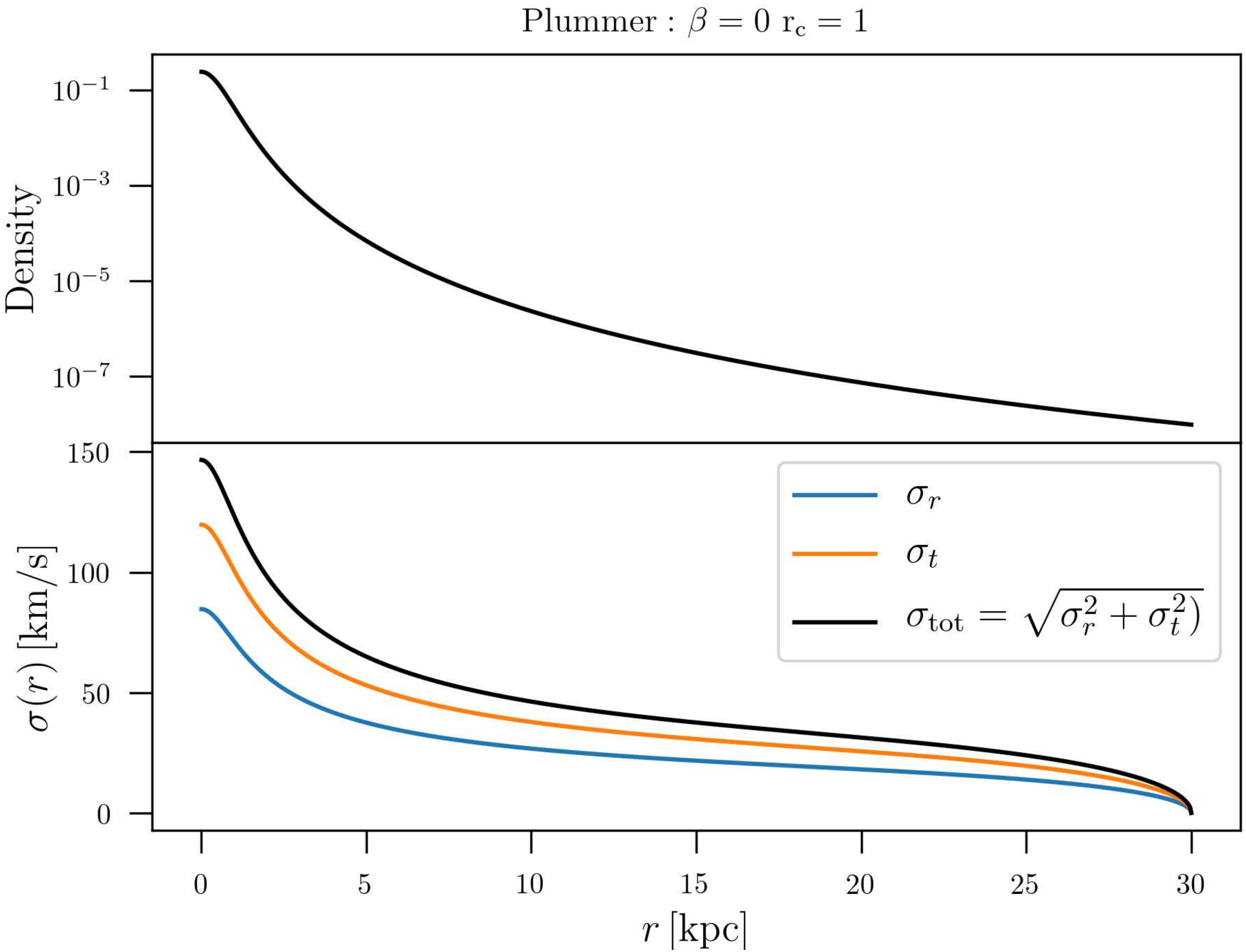
If the system has a constant anisotrpy parameter $\beta = cte$

$$\boxed{\sigma_r^2(r) = \frac{1}{r^{2\beta}\nu(r)}\int_r^\infty \mathrm{d}r' r'^{2\beta}\nu(r')\frac{\partial\Phi}{\partial r'} = \frac{G}{r^{2\beta}\nu(r)}\int_r^\infty \mathrm{d}r' r'^{2\beta-2}\nu(r')M(r')}$$

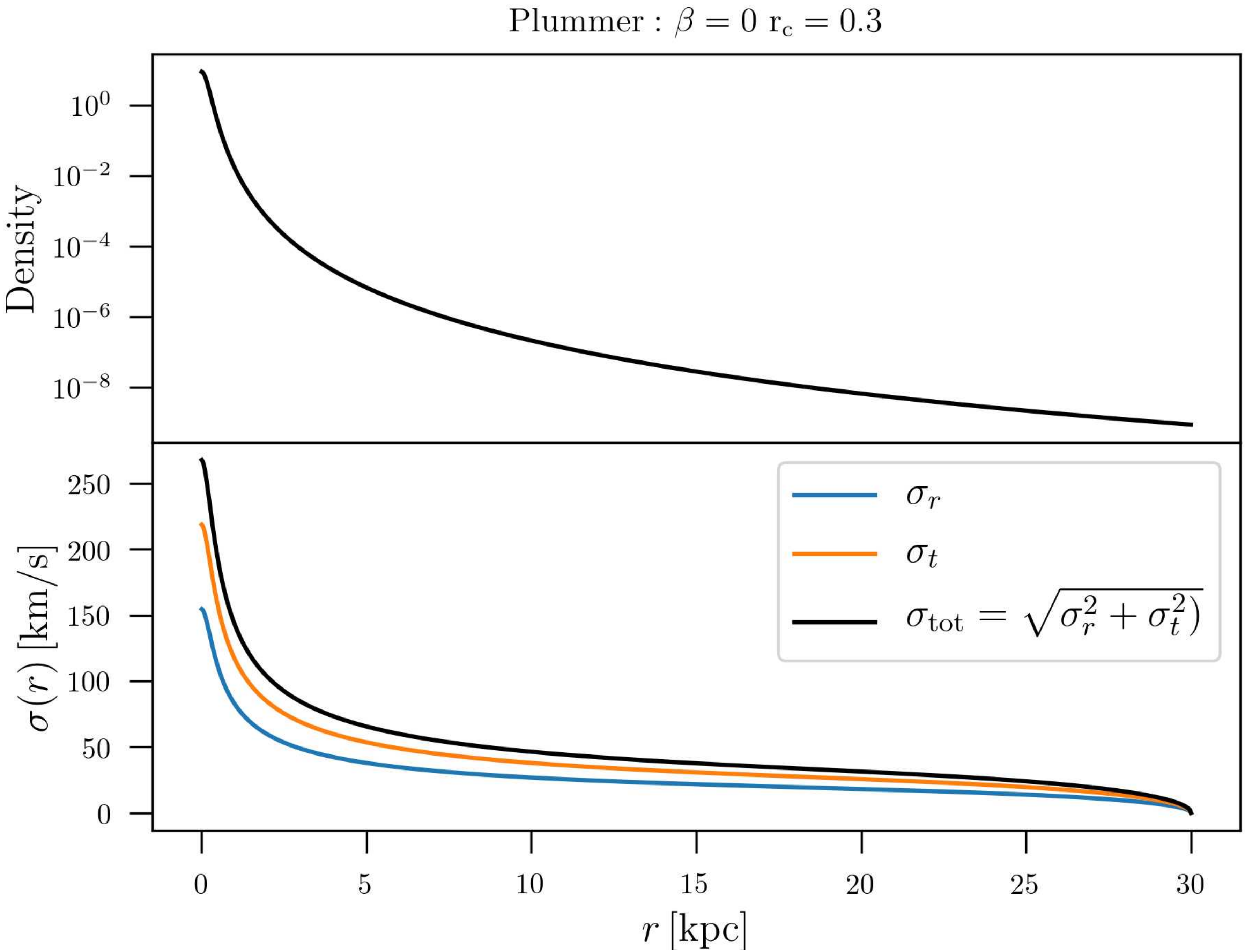
If the system is ergodic (isotropic in velocities) $\beta = 0$

$$\boxed{\sigma_r^2(r) = \frac{1}{\nu(r)}\int_r^\infty \mathrm{d}r' \nu(r')\frac{\partial\Phi}{\partial r'} = \frac{G}{\nu(r)}\int_r^\infty \mathrm{d}r' \frac{1}{r'^2}\nu(r')M(r')}$$

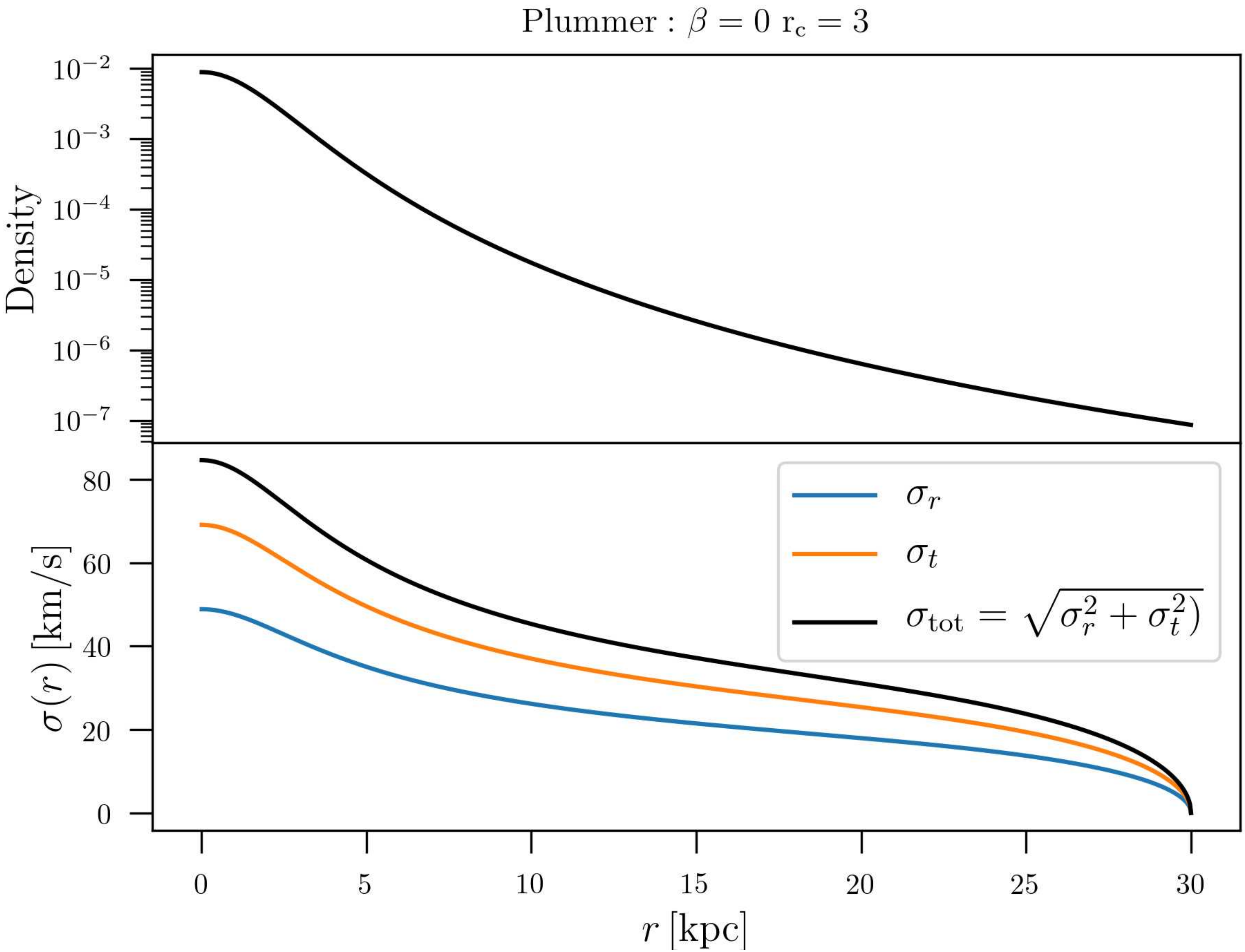
Play with the core radius R_c



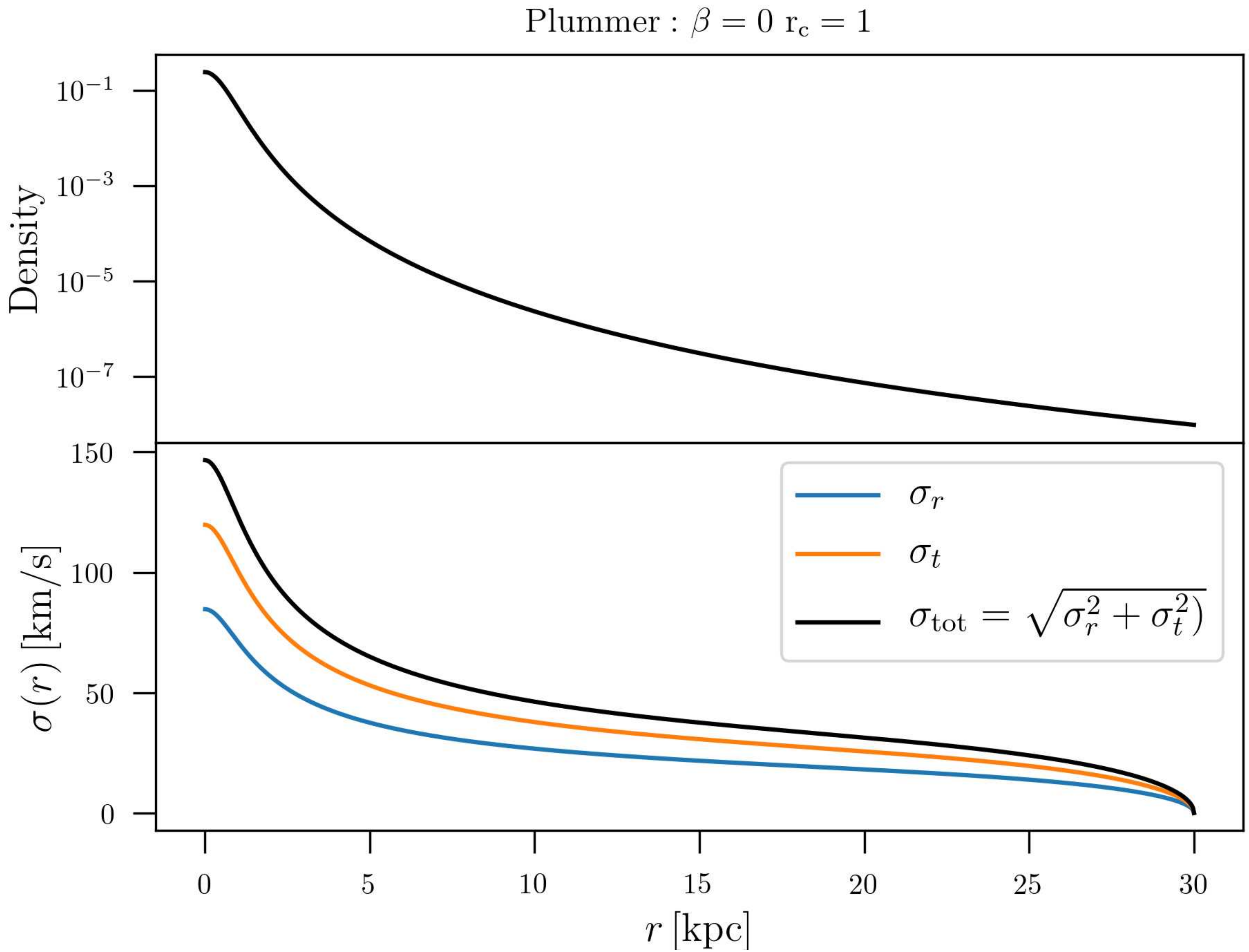
Play with the core radius R_c



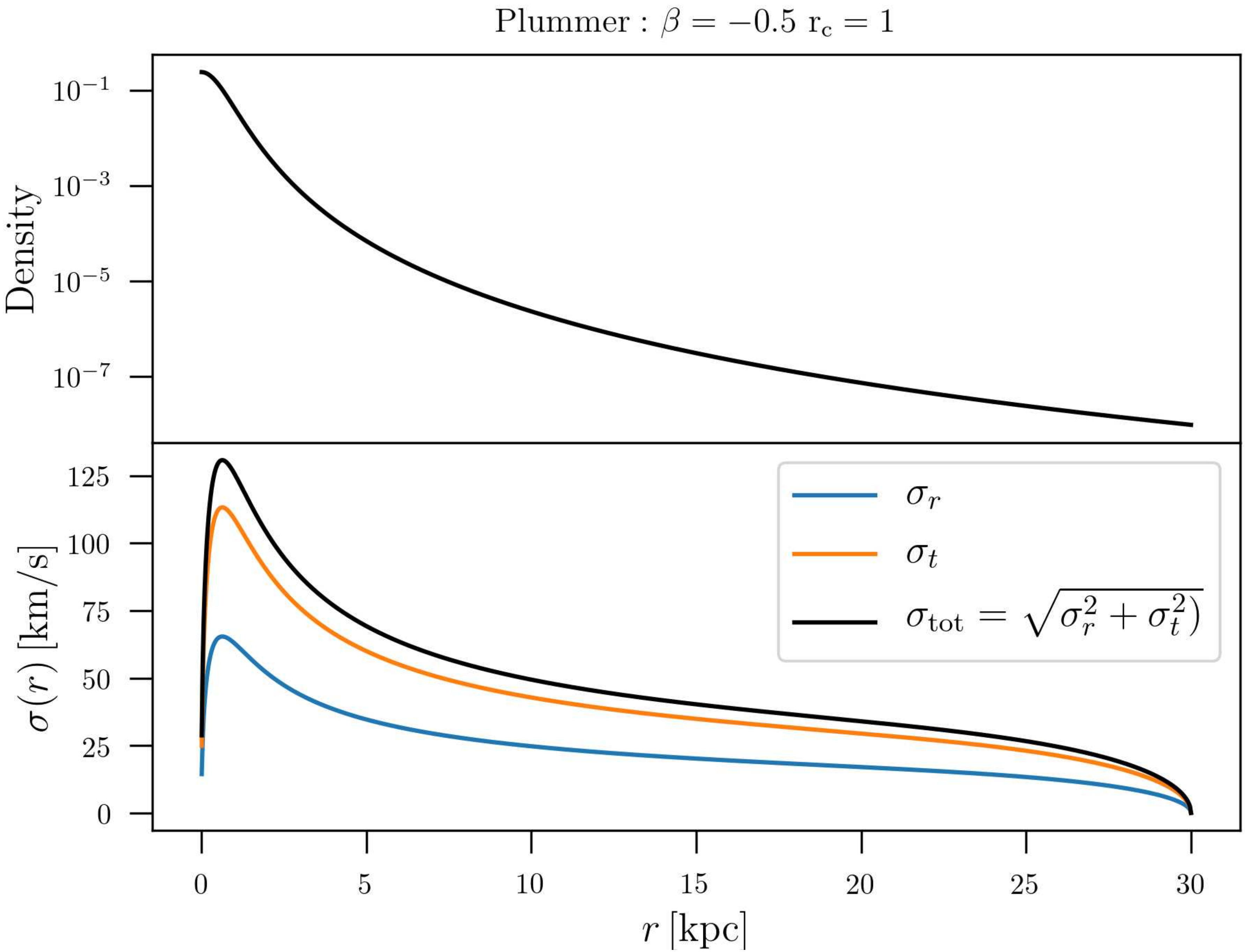
Play with the core radius R_c



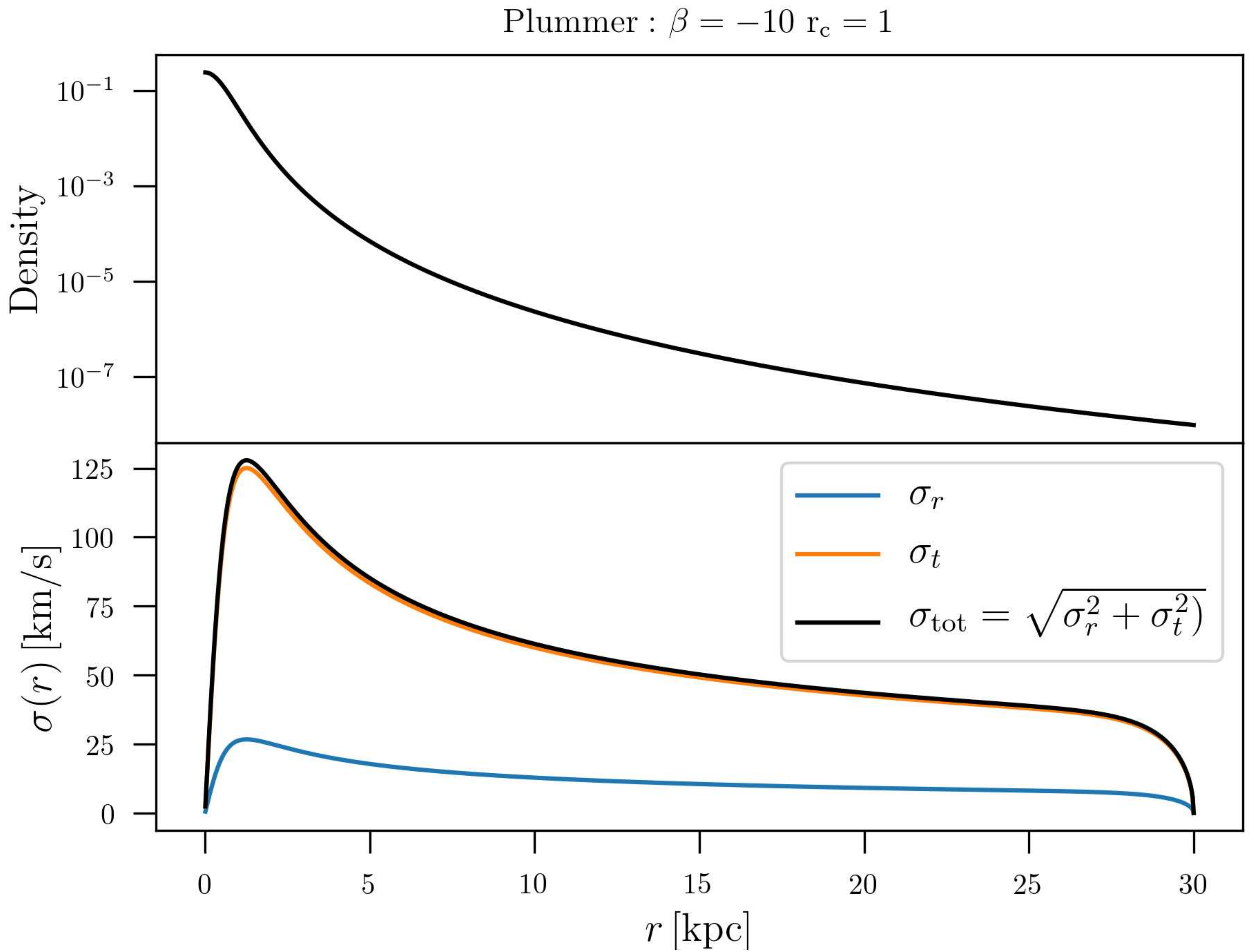
Play with the core radius R_c



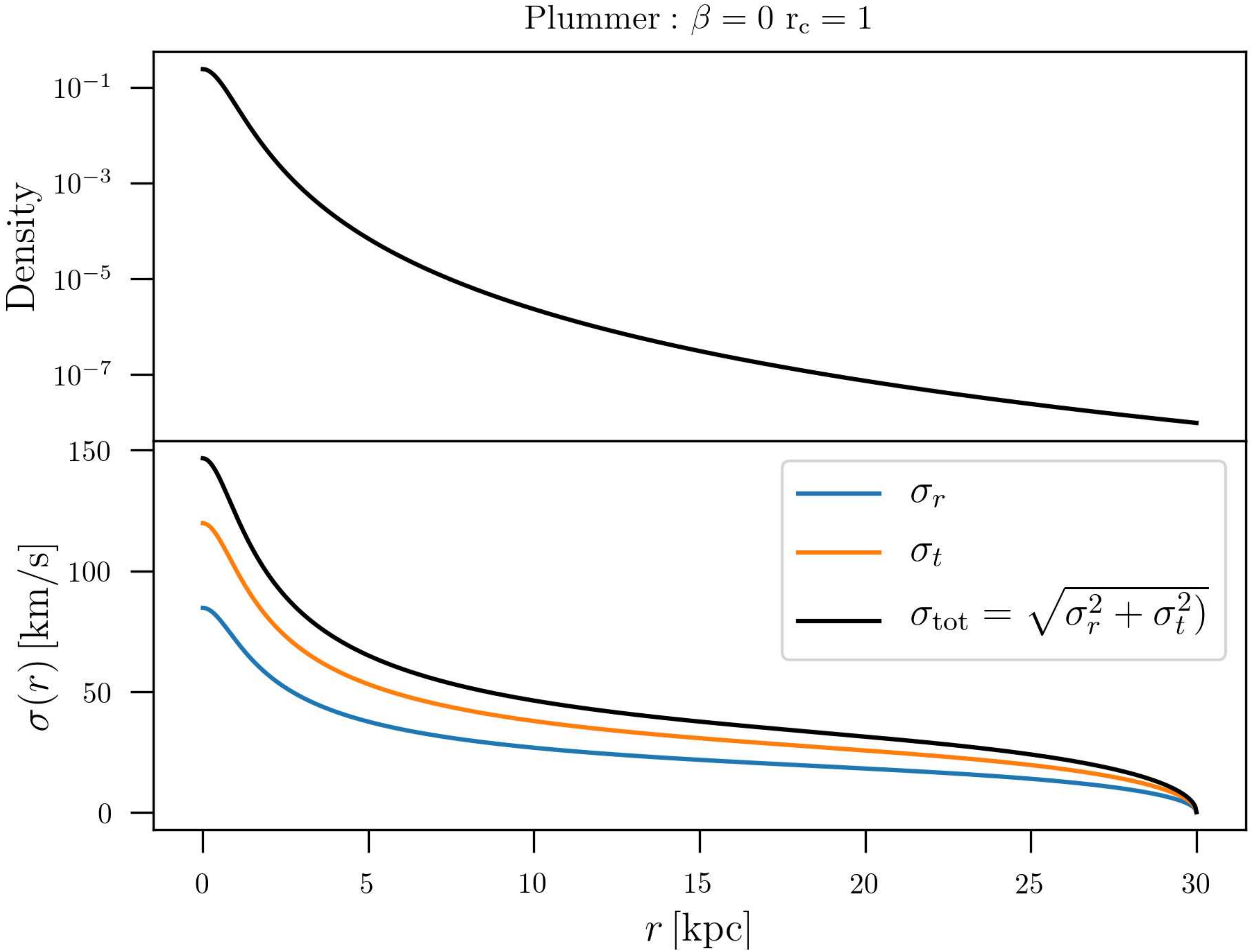
Play with the anisotropy parameter



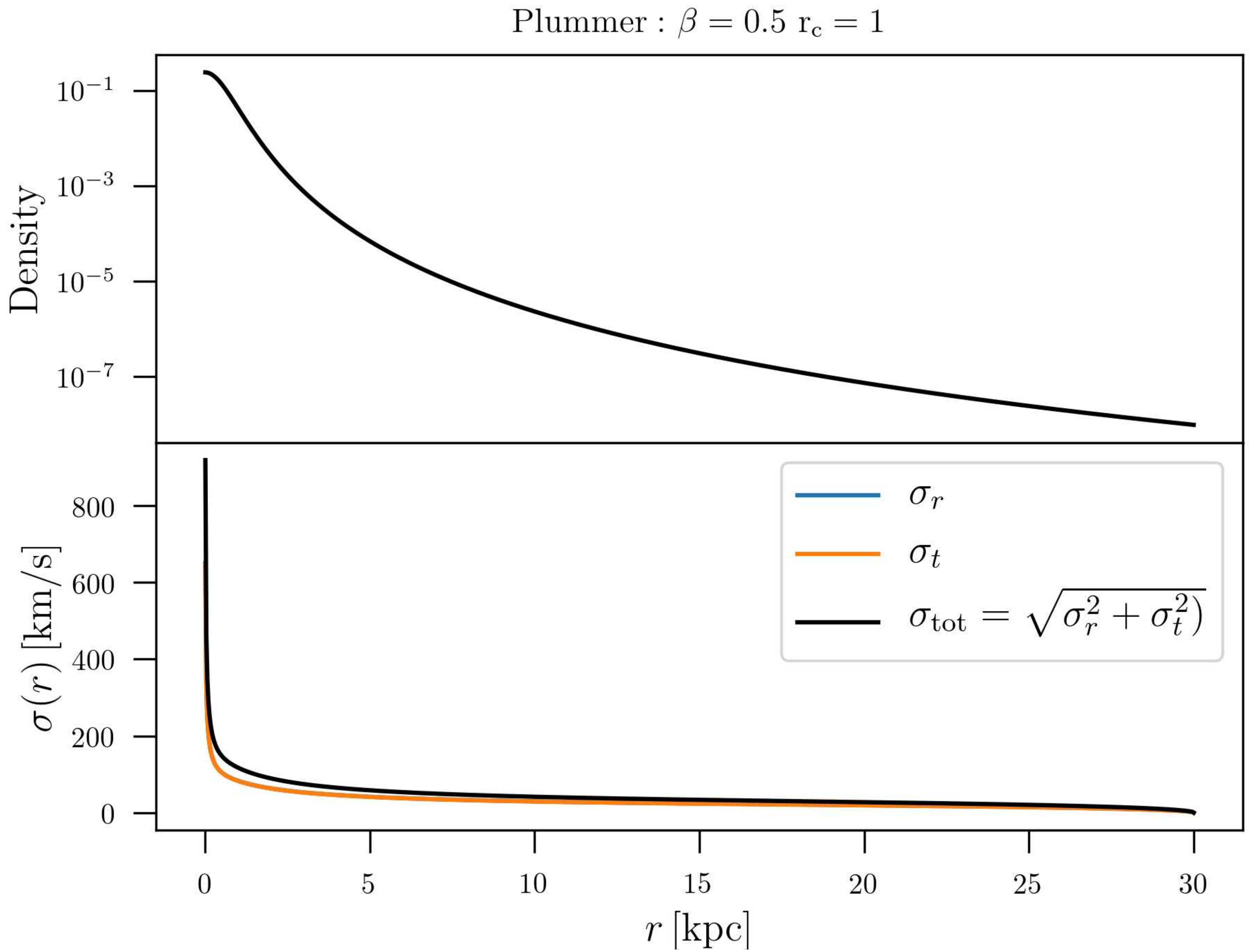
Play with the anisotropy parameter



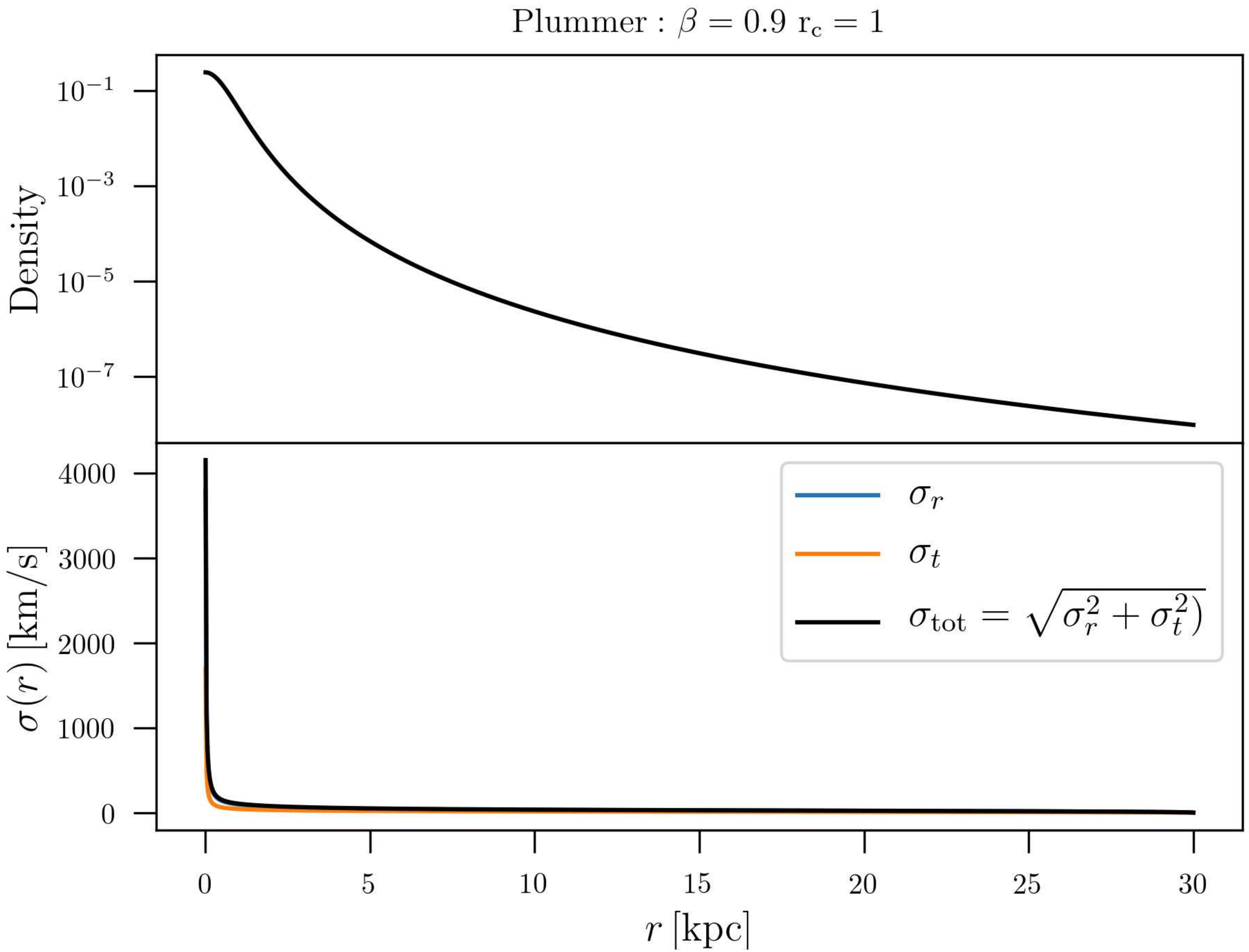
Play with the anisotropy parameter



Play with the anisotropy parameter

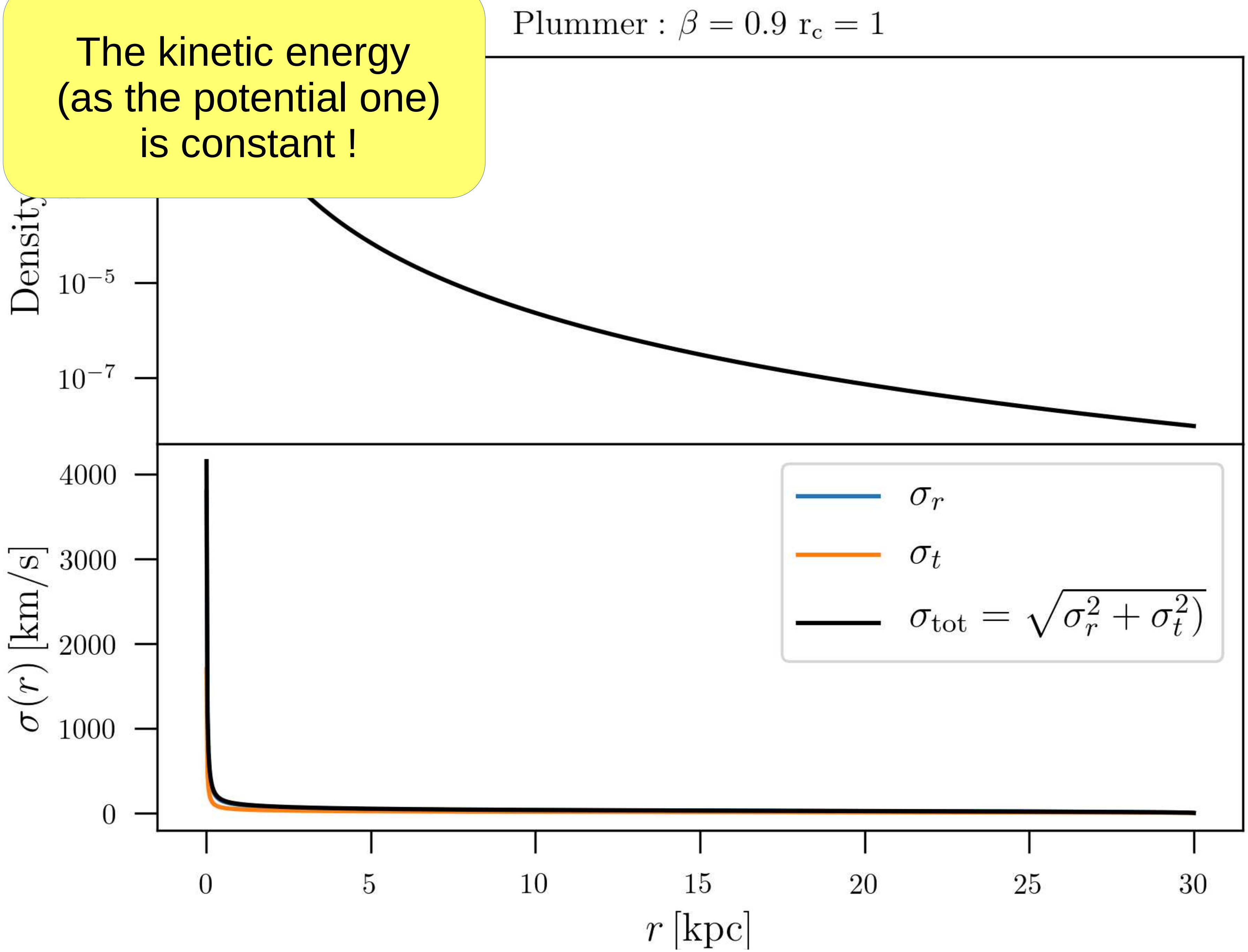


Play with the anisotropy parameter



Play with the anisotropy parameter

The kinetic energy
(as the potential one)
is constant !



Note on the pressure

For an ergodic system, defining

$$\boxed{P(\rho) = -\int_0^{\rho} d\rho' \, \rho' \, \frac{\partial \phi}{\partial \rho}(\rho')}$$

leads to $\dfrac{\vec{\nabla} P}{\rho} = -\vec{\nabla}\phi$

Comparing the Jeans equations with Euler one suggests

$P \equiv \rho \sigma^2$ but

$$\boxed{\rho \sigma^2(r) = \int_r^{\infty} dr' \, \rho(r') \, \frac{\partial \phi}{\partial r}}$$

So, is

$$P(\rho) = -\int_0^{\rho} d\rho' \, \rho' \, \frac{\partial \phi}{\partial \rho}(\rho') \stackrel{?}{=} P(r) = \int_r^{\infty} dr' \, \rho(r') \, \frac{\partial \phi}{\partial r}$$

$$P(\rho) = -\int_0^{\rho} d\rho' \, \rho' \, \frac{\partial\phi}{\partial\rho}(\rho') \qquad P(r) = \int_r^{\infty} dr' \, \rho(r') \, \frac{\partial\phi}{\partial r}$$

(1)

For a spherical system

$$\rho = \rho(r) \qquad \phi = \phi(r)$$

$$d\rho = \frac{d\rho}{\partial r} dr \qquad \frac{\partial\phi}{\partial\rho} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial\rho}$$

(1) becomes $-\int_{\infty}^{r} \cancel{\frac{\partial\rho}{\partial r}}' dr' \, \rho(r') \, \frac{\partial\phi}{\partial r}' \cancel{\frac{\partial r}{\partial\rho}}' = \int_r^{\infty} dr' \, \rho(r') \, \frac{\partial\phi}{\partial r}$

R $\phi(\infty) = 0$

#

Equilibria of collisionless systems

“Static” Jeans Equations for cylindrical systems

The Jeans equations for axisymmetric systems

Canonical momenta

$$\begin{cases} p_R = \dot{R} = v_R \\ p_\phi = R^2\dot{\phi} = R v_\phi \\ p_z = \dot{z} = v_z \end{cases}$$

The static Collisionless Boltzmann Equation, for axisymmetric systems

$$\cancel{\frac{\partial f}{\partial t}} + p_R\frac{\partial f}{\partial R} + \frac{p_\phi}{R^2}\cancel{\frac{\partial f}{\partial \phi}} + p_z\frac{\partial f}{\partial z} - \left(\frac{\partial \Phi}{\partial R} - \frac{p_\phi^2}{R^3}\right)\frac{\partial f}{\partial p_R} - \cancel{\frac{\partial \Phi}{\partial \phi}}\frac{\partial f}{\partial p_\phi} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial p_z} = 0$$

Zeroth order moment of the Jeans Equations if $f = f(H, L_z) \Rightarrow \overline{v_R^2} = \overline{v_z^2}, \overline{v_R} = \overline{v_z} = 0$

$$\overline{v_r^2} = \sigma_r^2 \; \overline{v_z^2} = \sigma_z^2$$

$$0 = 0$$

$$0 = 0$$

$$0 = 0$$

The Jeans equations for axisymmetric systems

Canonical momenta

$$\begin{cases} p_R = \dot{R} = v_R \\ p_\phi = R^2\dot{\phi} = Rv_\phi \\ p_z = \dot{z} = v_z \end{cases}$$

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$$\cancel{\frac{\partial f}{\partial t}} + p_R \frac{\partial f}{\partial R} + \cancel{\frac{p_\phi}{R^2}\frac{\partial f}{\partial \phi}} + p_z \frac{\partial f}{\partial z} - \left(\frac{\partial \Phi}{\partial R} - \frac{p_\phi^2}{R^3}\right)\frac{\partial f}{\partial p_R} - \cancel{\frac{\partial \Phi}{\partial \phi}}\frac{\partial f}{\partial p_\phi} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial p_z} = 0$$

First order moment of the Jeans Equations

$$\frac{\partial}{\partial R}\left(\nu\overline{v_R^2}\right) + \frac{\partial}{\partial z}\left(\nu\overline{v_R v_z}\right) + \nu\left(\frac{\overline{v_R^2} - \overline{v_\phi^2}}{R} + \frac{\partial \Phi}{\partial R}\right) = 0$$

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\nu\overline{v_R v_z}\right) + \frac{\partial}{\partial z}\left(\nu\overline{v_z^2}\right) + \nu\frac{\partial \Phi}{\partial z} = 0$$

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\nu\overline{v_R v_\phi}\right) + \frac{\partial}{\partial z}\left(\nu\overline{v_z v_\phi}\right) = 0$$

The Jeans equations for axisymmetric systems

Canonical momenta

$$\begin{cases} p_R = \dot{R} = v_R \\ p_\phi = R^2\dot{\phi} = Rv_\phi \\ p_z = \dot{z} = v_z \end{cases}$$

The static Collisionless Boltzmann Equation, for axisymmetric systems

$$\cancel{\frac{\partial f}{\partial t}} + p_R\frac{\partial f}{\partial R} + \frac{p_\phi}{R^2}\cancel{\frac{\partial f}{\partial \phi}} + p_z\frac{\partial f}{\partial z} - \left(\frac{\partial \Phi}{\partial R} - \frac{p_\phi^2}{R^3}\right)\frac{\partial f}{\partial p_R} - \cancel{\frac{\partial \Phi}{\partial \phi}}\frac{\partial f}{\partial p_\phi} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial p_z} = 0$$

First order moment of the Jeans Equations

if $f = f(H, L_z) \Rightarrow \overline{v_R^2} = \overline{v_z^2}, \overline{v_R} = \overline{v_z} = 0$

$\overline{v_r^2} = \sigma_r^2 \; \overline{v_z^2} = \sigma_z^2$

$$\frac{\partial}{\partial R}\left(\nu\overline{v_R^2}\right) + \nu\left(\frac{\overline{v_R^2} - \overline{v_\phi^2}}{R} + \frac{\partial \Phi}{\partial R}\right) = 0$$

$$\frac{\partial}{\partial z}\left(\nu\overline{v_z^2}\right) + \nu\frac{\partial \Phi}{\partial z} = 0 \qquad \Rightarrow \qquad \boxed{\overline{v_R^2}(R,z) = \overline{v_z^2}(R,z) = \frac{1}{\nu(R,z)}\int_z^\infty \mathrm{d}z'\nu(R,z')\frac{\partial \Phi}{\partial z'}}$$

$$0 = 0 \qquad \Rightarrow \qquad \boxed{\overline{v_\phi^2}(R,z) = \overline{v_R^2} + \frac{R}{\nu(R,z)}\frac{\partial}{\partial R}\left(\nu\overline{v_R^2}\right) + R\frac{\partial \Phi}{\partial R}}$$

Jeans equations for axisymmetric systems

$$\overline{v_z^2} = \frac{1}{\nu} \int_z^\infty dz' \, \nu(R, z') \frac{\partial \phi}{\partial z'}$$

<u>Note</u> $\overline{v_z^2} = \sigma_z^2 = \overline{v_R^2} = \sigma_R^2$ as $f = f(H, L_z)$

$$\overline{v_\phi^2}(R,z) = \sigma_R^2 + \frac{R}{\nu} \frac{\partial}{\partial R}\left(\nu \sigma_R^2\right) + R \frac{\partial \phi}{\partial R}$$

Interpretation

$$\boxed{\overline{V_\phi^2}(R,z) = \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right) + R\frac{\partial\phi}{\partial R}}$$

In the plane $z=0$

- $R\dfrac{\partial\phi}{\partial R} = V_c^2$
- $\overline{V_\phi^2} = \sigma_\phi^2 + \overline{V_\phi}^2$

$$\boxed{\overline{V_\phi}^2 = V_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)}$$

1 Equation, 2 Unknowns $\overline{V_\phi}$ σ_ϕ

This equation involves different energies

Interpretation

$$\boxed{\overline{V_\phi}^2 = V_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)}$$

1. if $\sigma_\phi = \sigma_R = 0$ $\quad (\Rightarrow \sigma_z = 0)$ as $\sigma_R = \sigma_z$ $\quad$! ok if $\nu \sim \delta(z)$ = razor thin disk

$$\boxed{\overline{V_\phi}^2 = V_c^2}$$

The mean azimuthal velocity is the circular velocity

The disk is "super cold"

$$\sigma_R = \sigma_z = \sigma_\phi = 0$$

2. if $\sigma_R = 0,\ \sigma_\phi \neq 0$ $\quad (\Rightarrow \sigma_z = 0)$ $\quad$! ok if $\nu \sim \delta(z)$ = razor thin disk

$$\boxed{\overline{V_\phi}^2 = V_c^2 - \sigma_\phi^2}$$

But $\sigma_r = 0 \Rightarrow$ only circular orbits

1 $\overline{V_\phi}^2 = V_c^2 \Rightarrow \sigma_\phi = 0$ ⚠

2 $\overline{V_\phi}^2 = 0 \Rightarrow$ counter rotating disk with

$$\overline{V_\phi} = \frac{1}{2}\left(V_c - V_c\right) = 0$$

$$\sigma_\phi^2 = \frac{1}{2}\left(V_c^2 + V_c^2\right) = V_c^2$$

Interpretation

$$\boxed{\overline{V}_\phi^{\,2} = V_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)}$$

3. if $\sigma_R = \sigma_\phi \neq 0$ ("Ergodic")

$$\overline{V}_\phi^{\,2} = R\frac{\partial\phi}{\partial R} + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)$$

$$\frac{1}{R}\overline{V}_\phi^{\,2} = \frac{\partial\phi}{\partial R} + \frac{1}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)$$

$$\boxed{\underbrace{\frac{1}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)}_{\sim \frac{\vec{\nabla}P}{\rho}\ \text{"pressure" force}} = \underbrace{-\frac{\partial\phi}{\partial R}}_{\vec{F}_{grav}\ \text{gravit. force}} + \underbrace{\frac{\overline{V}_\phi^{\,2}}{R}}_{\text{centrifugal force}}}$$

Equilibrium in the rotating frame $\Omega = \frac{\bar{V}_\phi}{R}$

$F_c = \Omega^2 R$ $\quad \Omega = \frac{V}{R}$

$= \frac{V^2}{R}$

Interpretation

$$\boxed{\overline{V_\phi}^2 = V_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu \sigma_R^2\right)}$$

4. if $\sigma_\phi = 0$, $\sigma_r \neq 0$ (radial orbits)

$$0 = V_c^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu \sigma_R^2\right)$$

$$\frac{1}{\nu}\frac{\partial}{\partial R}\left(\nu \sigma_R^2\right) + \frac{\sigma_R^2}{R} = -\frac{\partial \phi}{\partial R}$$

Nearly identical to the spherical case.

$$\frac{1}{\nu}\frac{\partial}{\partial r}\left(\nu \sigma_r^2\right) + \frac{2\sigma_r^2}{r} = \frac{\partial \phi}{\partial r}$$

How to close the equation? i.e., chose σ_ϕ ?

- Assume that stars are near circular orbits

$$\begin{cases} \ddot{x} = -\varkappa^2 x \\ \ddot{y} = -\varkappa^2 y \end{cases}$$

oscillations around the guiding center

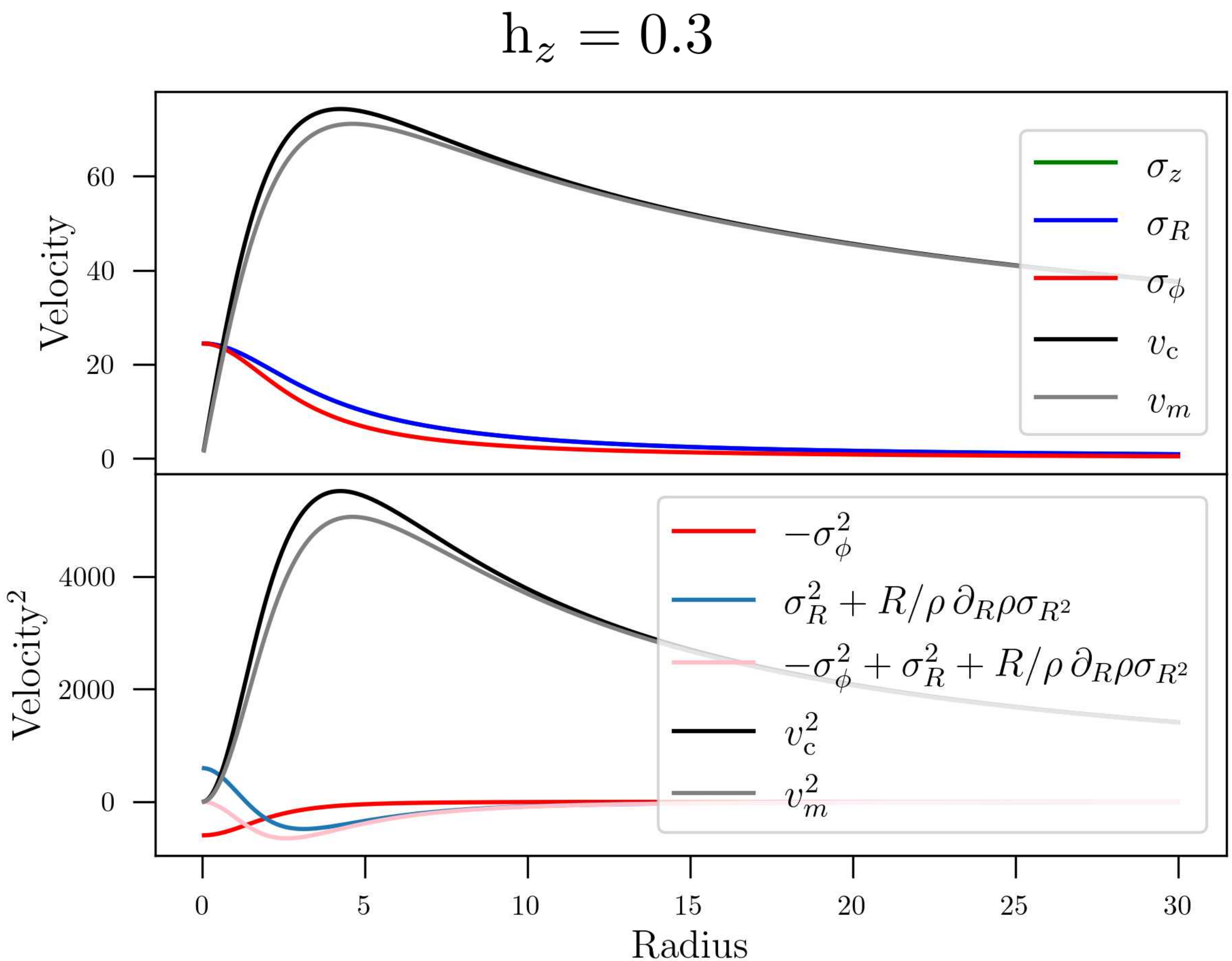
$$\begin{cases} \dot{x}(t) = -X\varkappa \sin(\varkappa t + \alpha) \\ \dot{y}(t) = -Y\varkappa \cos(\varkappa t + \alpha) \end{cases} \qquad Y = \frac{2\Omega_s}{\varkappa} X$$

$$\sigma_r^2 \equiv \sigma_x^2 = \frac{1}{2\pi} \int_0^{\frac{2\pi}{\varkappa}} X^2 \varkappa^2 \sin^2(\varkappa t + \alpha)\, dt = \frac{X^2 \varkappa^2}{2}$$

$$\sigma_\phi^2 \equiv \sigma_Y^2 = \frac{1}{2\pi} \int_0^{\frac{2\pi}{\varkappa}} Y^2 \varkappa^2 \cos^2(\varkappa t + \alpha)\, dt = \frac{Y^2 \varkappa^2}{2}$$

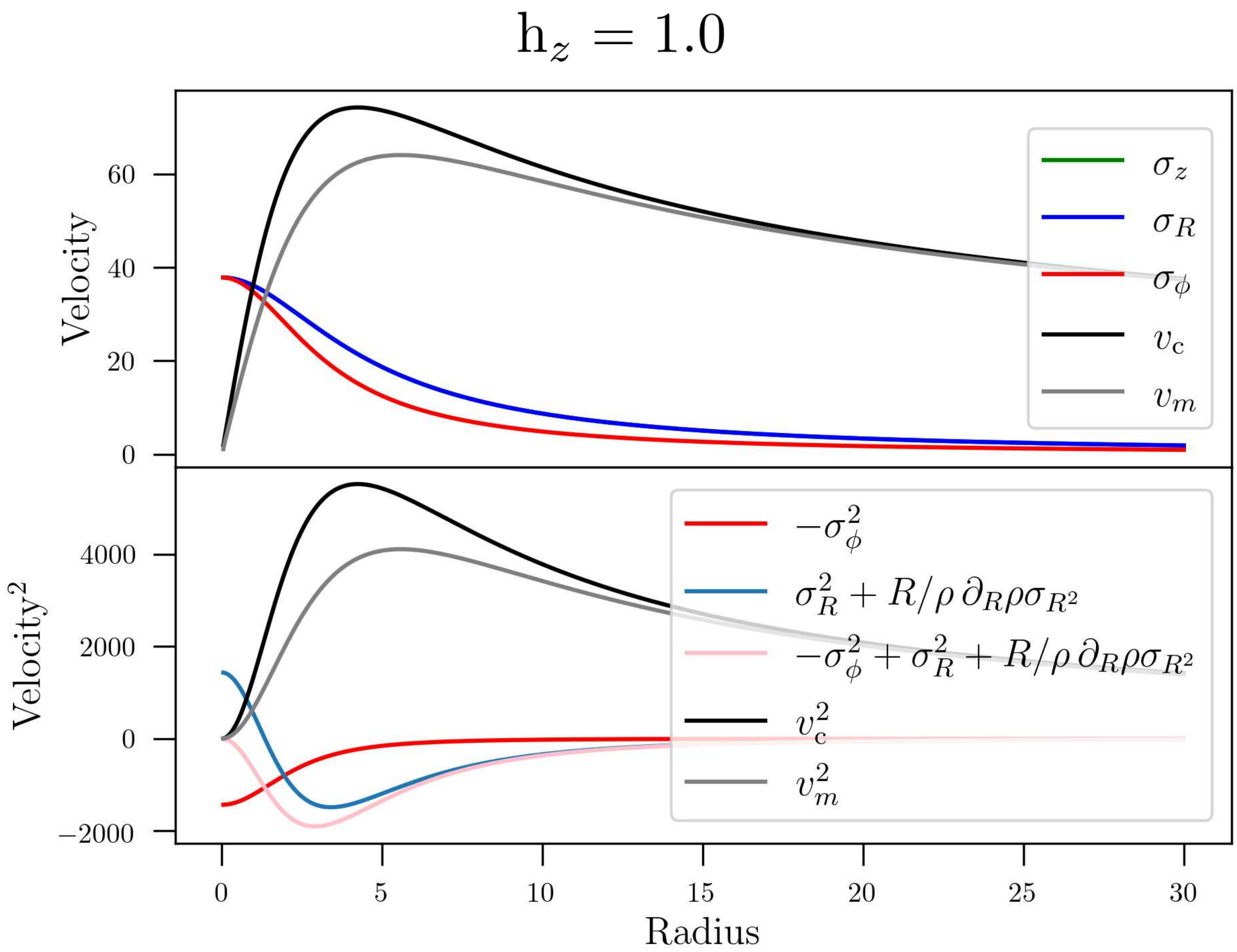
thus $\boxed{\sigma_\phi^2 = \frac{\varkappa^2}{4\Omega_s^2} \sigma_r^2}$

Jeans Moments and rotation curve for a Miyamoto-Nagai disk



$$\sigma_z^2 = \frac{1}{\nu}\int_z^\infty \mathrm{d}z'\nu\frac{\partial\Phi}{\partial z'} \qquad \sigma_R^2 = \sigma_z^2 \qquad \frac{\sigma_\phi^2}{\sigma_R^2} = \frac{\kappa^2}{4\Omega^2} \qquad \overline{v_\phi}^2 = v_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)$$

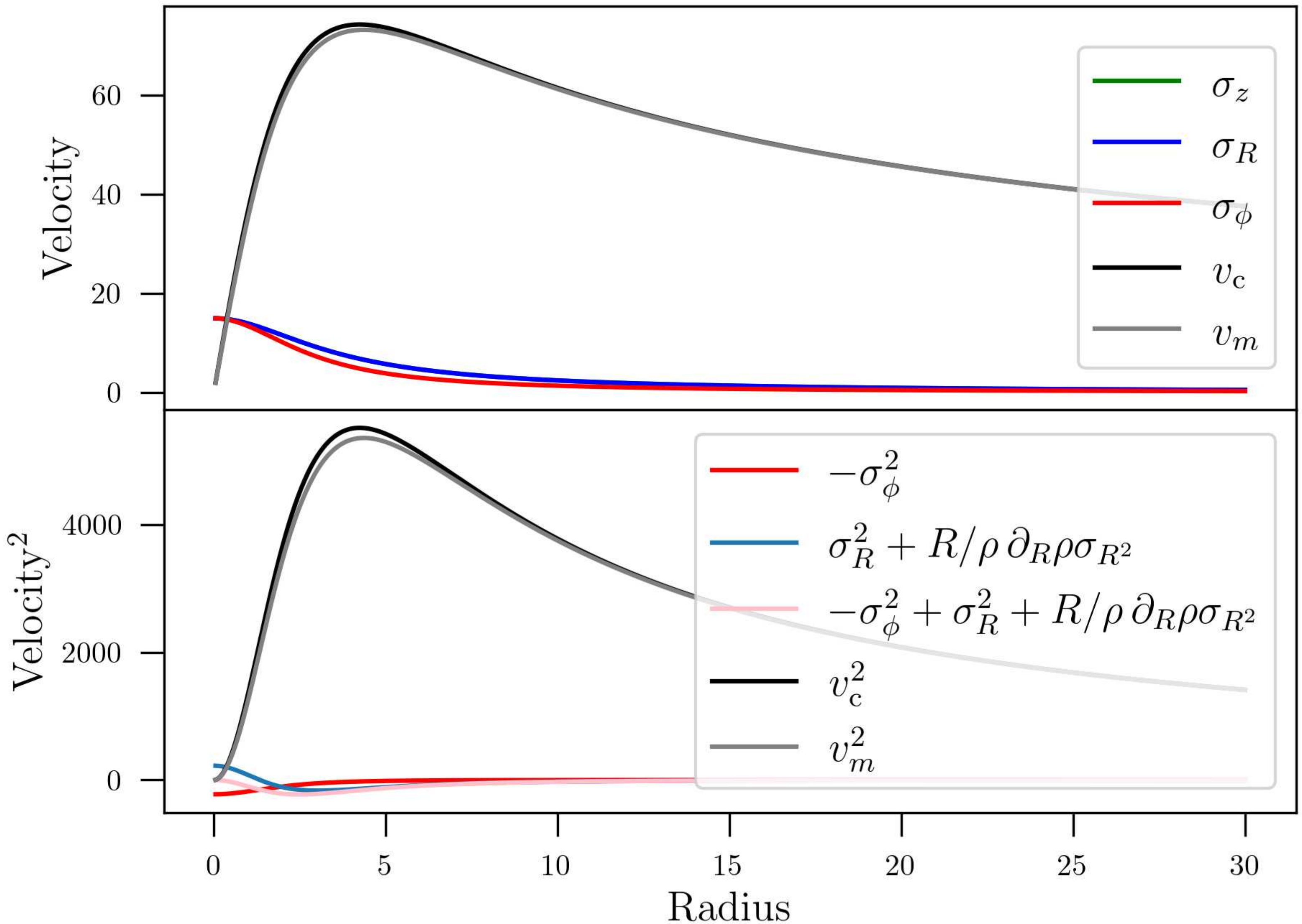
Jeans Moments and rotation curve for a Miyamoto-Nagai disk



$$\sigma_z^2 = \frac{1}{\nu}\int_z^\infty \mathrm{d}z'\nu\frac{\partial\Phi}{\partial z'} \qquad \sigma_R^2 = \sigma_z^2 \qquad \frac{\sigma_\phi^2}{\sigma_R^2} = \frac{\kappa^2}{4\Omega^2} \qquad \overline{v_\phi}^2 = v_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)$$

Jeans Moments and rotation curve for a Miyamoto-Nagai disk

$\mathrm{h}_z = 0.1$



$$\sigma_z^2 = \frac{1}{\nu}\int_z^\infty \mathrm{d}z'\,\nu\frac{\partial\Phi}{\partial z'} \qquad \sigma_R^2 = \sigma_z^2 \qquad \frac{\sigma_\phi^2}{\sigma_R^2} = \frac{\kappa^2}{4\Omega^2} \qquad \overline{v_\phi}^2 = v_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)$$

The End