EPFL Automne 2022
Algebre Lineaire Avancee, MATH-110

Serie 10

Exercice 1

1. $\phi(x,y,z,t) = M\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x+2y+3z+4t \\ 2x+3y+4z+t \\ 3x+4y+z+2t \end{pmatrix}$ $\forall (x,y,z,t) \in \mathbb{Q}$

2. On cherche une base de $ker(\phi)$: On resout

$$\phi(x,y,z,t) = \begin{pmatrix} x+2y+3z+4t \\ 2x+3y+4z+t \\ 3x+4y+z+2t \end{pmatrix} = (0,0,0)$$

et on obtient $ker(\phi) = \{(11t, -9t, t, t) : t \in \mathbb{Q}\}$. C'est evidemment une famille libre et generatrice et donc une base de $ker(\phi)$

Image : On raisonne sur les lignes de la matrice M qui forment une famille generatrice de $im(\phi)$. Il suffit de montrer que $\{(1,2,3,4),(2,3,4,1),(3,4,1,2)\}$ est une famille livre et ainsi celle ci constituera une base de l'image.

3. Il faut trouver 2 bases B_3 et B_4 dans lesquelles $mat_{B_3,B_4} = Id_{3\times 4}$. On remarque

$$\phi(0,1,0,0) = (2,3,4), \phi(0,0,1,0) = (3,4,1), \phi(11,-9,1,1) = (0,0,0)$$

et on pose

$$B_3 = \{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}\}, B_4 = \{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 11 \\ -9 \\ 1 \\ 1 \end{pmatrix}\}$$

On a donc

$$mat_{B_3,B_3^0} = \begin{pmatrix} -13/4 & 5/2 & -1/4 \\ 5/2 & -2 & 1/2 \\ -1/4 & 1/2 & -1/4 \end{pmatrix}, mat_{B_4^0,B_4} = \begin{pmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercice 2

1. We will denote the elements of the canonical basis $\mathscr{B}^0$ by $e_1 = (1,0)$ and $e_2 = (0,1)$. Observe that
$$\varphi(e_1) = (\frac{1}{3}, \frac{-1}{3})$$
$$\varphi(e_2) = (\frac{4}{3}, \frac{5}{3})$$
Therefore we have
$$\mathrm{Mat}_{\mathscr{B}^0\mathscr{B}^0} = \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \\ \frac{-1}{3} & \frac{5}{3} \end{pmatrix}$$

2. Obsereve that if $\mathrm{car}(K) \neq 3$ f_1 and f_2 are linearly independent. By definition
$$\begin{cases} 2\lambda_1 = \lambda_2 \\ 2\lambda_2 = -\lambda_1 \end{cases}$$
$$\begin{cases} 2\lambda_1 = \lambda_2 \\ 3\lambda_2 = 0 \end{cases}.$$
If $\mathrm{car}(K) \neq 3$ then $\lambda_1 = \lambda_2 = 0$ and the vectors are linearly independent. If $\mathrm{car}(K) = 3$ note that the system has solution for non-zero λ_2 therefore the vectors are not linearly independent.
We will now compute the matrix of φ in the basis of $\mathscr{B}$ denoted by $\mathrm{Mat}_{\mathscr{B}\mathscr{B}}$ using the formula of changing of basis. For the formula of changing of basis we have
$$N = \mathrm{Mat}_{\mathscr{B}\mathscr{B}} = \mathrm{Id}_{\mathscr{B}\mathscr{B}^0}\mathrm{Mat}_{\mathscr{B}^0\mathscr{B}^0}\mathrm{Id}_{\mathscr{B}^0\mathscr{B}}.$$
where by Id_{AB} we denote the matrix of chaning of basis from A to B.
First we compute $\mathrm{Id}_{\mathscr{B}\mathscr{B}^0}$. Observe that
$$(e_1) = \frac{2}{3}f_1 - \frac{1}{3}f_2$$
$$(e_2) = \frac{2}{3}f_2 - \frac{1}{3}f_1$$
Therefore
$$\mathrm{Id}_{\mathscr{B}\mathscr{B}^0} = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix}.$$
Then we compute $\mathrm{Id}_{\mathscr{B}\mathscr{B}^0}$. Observe that $f_1 = 2e_1 + e_2$ and $f_2 = e_1 + 2e_2$. Then
$$\mathrm{Id}_{\mathscr{B}^0\mathscr{B}} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Therefore we have

$$\text{Mat}_{\mathscr{B}\mathscr{B}} = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \\ \frac{-1}{3} & \frac{5}{3} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

$$N = \text{Mat}_{\mathscr{B}\mathscr{B}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We know compute $\text{Mat}_{\mathscr{B}\mathscr{B}}$ directly. In order to do this one has to compute

$$\varphi(f_1) = (2,1) = 1f_1 = 0f_2$$
$$\varphi(f_2) = (3,3) = f_1 + f_2.$$

These coefficient form the columns of the matrix $\text{Mat}_{\mathscr{B}\mathscr{B}}$ which is therefore given by

$$N = \text{Mat}_{\mathscr{B}\mathscr{B}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This confirm our computation when using the formula of changing of basis.

3. Observe that by simple computations one can see that taking the n-th power of the matrix N gives us
$$N = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

4. Observe that since $C^{-1}C = \text{Id}$ we have
$$(C.U.C^{-1})^n = C.U.C^{-1}C.U.C^{-1}C.U.C^{-1} \cdots C.U.C^{-1}C.U.C^{-1} = CUU \cdots UC^{-1}C = C.U^n.C^{-1}.$$

5. We know from the theory of the course that $Id_{\mathscr{B}\mathscr{B}^0}$ is invertible and
$$Id_{\mathscr{B}\mathscr{B}^0} = Id^{-1}_{\mathscr{B}\mathscr{B}^0}$$
Therefore for the formula of changing of basis
$$M = \text{Id}_{\mathscr{B}^0\mathscr{B}}\text{Mat}_{\mathscr{B}\mathscr{B}}\text{Id}^{-1}_{\mathscr{B}^0\mathscr{B}}.$$
$$M = \text{Id}_{\mathscr{B}^0\mathscr{B}}N\text{Id}^{-1}_{\mathscr{B}^0\mathscr{B}}.$$
So for the formula of point 4 we have
$$M^n = \text{Id}_{\mathscr{B}^0\mathscr{B}}N^n\text{Id}^{-1}_{\mathscr{B}^0\mathscr{B}}.$$

Exercice 3

On a

$$((M.N)^\top)_{ij} = (MN)_{ji} = \sum_{k=1}^{d'} M_{jk}N_{ki} = \sum_{k=1}^{d'} (N^\top)_{ik}(N^\top M^\top)_{ij} = (N^\top M^\top)_{ij}$$

Ainsi les composantes sont egales et on conclut.

Exercice 4

From now on the matrix multiplication $A.B$ will be just denoted by AB.

1. To prove that the application $[A.]$ is linear it is sufficient to show that $[A.](N_1 + \lambda N_2) = [A.](N_1) + \lambda [A.](N_2)$ for any $\lambda \in K$ and $N_1, N_2 \in M_2(K)$. This follows by the properties of multiplication of matrices.
2. Consider the canonical basis $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,$E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then we have

   $$AE_{11} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix} = a_{11}E_{11} + a_{21}E_{21}.$$

   Similarly we can compute

   $$\begin{aligned} AE_{12} &= a_{11}E_{12} + a_{21}E_{22}, \\ AE_{21} &= a_{12}E_{11} + a_{22}E_{21}, \\ AE_{22} &= a_{12}E_{12} + a_{22}E_{22} \end{aligned}$$

   .

   Therefore we can write $[A.]$ in matrix form with respect to the canonical basis as follows

   $$\mathcal{A} = \begin{pmatrix} a_{11} & 0 & a_{12} & 0 \\ 0 & a_{11} & 0 & a_{12} \\ a_{21} & 0 & a_{22} & 0 \\ 0 & a_{21} & 0 & a_{22} \end{pmatrix}.$$

3. This is equivalent to prove that

$$\det(A) \neq 0 \iff \det(\mathcal{A}) \neq 0.$$

Obsereve that

$$\det(\mathcal{A}) = a_{11} \det \begin{pmatrix} a_{11} & 0 & a_{12} \\ 0 & a_{22} & 0 \\ a_{21} & 0 & a_{22} \end{pmatrix} + a_{12} \det \begin{pmatrix} 0 & a_{11} & a_{12} \\ a_{21} & 0 & 0 \\ 0 & a_{21} & a_{22} \end{pmatrix} \neq 0$$

$$\Updownarrow$$

$$a_{11}a_{22} \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - a_{12}a_{21} \begin{pmatrix} a_1 1 & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq 0$$

$$\Updownarrow$$

$$a_{11}a_{22} \det(A) - a_{12}a_{21} \det(A) \neq 0$$

$$\Updownarrow$$

$$\det(A)^2 \neq 0$$

$$\Updownarrow$$

$$\det(A) \neq 0$$

4. We would lif to prove that $A^2 = 0 \iff \text{Im}[A.] \subset \ker[A.]$.

   We will use the following notation. A matrix $M = \begin{pmatrix} m_1 1 & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ will be denoted in vector form as $m = \begin{pmatrix} m_1 1 \\ m_{12} \\ m_{21} \\ m_{22} \end{pmatrix}$ (and we say M is the matrix form of m).

   We will first prove this implication ($\implies$) Suppose $A^2 = 0$ and let $v \in \text{Im}[A.]$ then there exists w such that $v = [A.]w$. This implies that in matrix form $V = AW$. Since $A^2 = 0$ we have $AV = AAw = 0$. This can be written as $[A.][A.]w = 0$. Therefore we can conclude $v = [A.]w \in \ker[A.]$.

   For the other implication. Suppose $Im[A.] \subset \ker[A.]$. We know that a, the vector form of the matrix A is in $Im[A.]$ (since $[A.]\text{Id} = a$). Therefore $[A.]A = A^2 = 0$.

5. Denote by Φ the map described in point 5. It is linear since $\Phi(A + \lambda B) = [A.] + \lambda[B.]$ for every $A, B \in M_2(K)$ and $\lambda \in K$. We let the reader prove the details of this. This map it is injecteive since

$$\Phi(A) = 0 \iff \mathcal{A} = 0 \iff A = 0.$$

Exercice 5

On demontre le cas general (dans l'exercice de la serie $d = 4$)

1. Let $e_1, e_2, \dots e_d$ be the canonical basis of K^2. Recall that we have computed φ^j in the previous Series 8 exercise 6. From there it is easy to see that $\varphi^{j-1}(e_1) = e_j$ for $j \in \{2, .., d\}$. Therefore $\{\mathbf{e}_1, \varphi(\mathbf{e}_1), \cdots, \varphi^{d-1}(\mathbf{e}_1)\} = e_1, e_2, \dots e_d$ is a basis for K^d. We also know that $\varphi^d(e_1) = (-b_0, -b_1, \dots, -b_d)$

2. It is sufficient to prove this is true on all the elements of the basis$e_1, e_2, \dots e_d$ of K^d. For e_1 we know that $\{\mathbf{e}_1, \varphi(\mathbf{e}_1), \cdots, \varphi^{d-1}(\mathbf{e}_1)\}$ form a basis therefore $\varphi^d(e_1)$ is a linear combination of them. Thus there exist $a_0 \dots a_d$ such that

$$0 = \varphi^d(e_1) + a_{d-1}(e_1)\varphi^{d-1}(e_1) + \cdots + a_1.\varphi(e_1) + a_0 e_1.$$

   Now,we want to prove this is true for all the other elements of the basis. We know that $e_j = \varphi^{j-1}(e_1)$. And using the a_i of the previous equation one can see the following

$$\varphi^d(e_j) + a_{d-1}(e_j)\varphi^{d-1}(e_j) + \cdots + a_1.\varphi(e_j) + a_0 e_j$$
$$\|$$
$$\varphi^d(\varphi^{j-1}(e_1)) + a_{d-1}(\varphi^{j-1}(e_1))\varphi^{d-1}(\varphi^{j-1}(e_1)) + \cdots + a_1.\varphi(e_j) + a_0\varphi^{j-1}(e_1)$$
$$\|$$
$$\varphi^{j-1}(\varphi^d(e_1) + a_{d-1}(e_1)\varphi^{d-1}(e_1) + \cdots + a_1.\varphi(e_1) + a_0 e_1)$$
$$\|$$
$$0$$

   Where in the last equation we used the linearity of φ and the fact that we knew $\varphi^d(e_1) + a_{d-1}(e_1)\varphi^{d-1}(e_1) + \cdots + a_1.\varphi(e_1) + a_0 e_1 = 0$.

3. This point follows from the previous one and from the fact that M is the matric representation of φ.

Exercice 6

1. We let to the reader to prove that indeed $\text{tr}(M + \lambda N) = \text{tr}(A) + \lambda \text{tr}(N)$ for any $M, N \in M_d(K)$ and $\lambda \in K$.

2. We will denote by $Diag(M) = (m_{11}, m_{22}, \dots, m_{dd})$ the vector given by the elements in the diagonal of the matrix M. Note that we can redefine the trace of a matrix as $\mathrm{tr}(M) = \sum_{m \in Diag(M)} m$.
   Then, since $Diag(M) = Diag(M^t)$ it follows that $\mathrm{tr}(M) = \mathrm{tr}(M^t)$. Moreover since the diaginal elements of MN are of the form $k_{ii} = \sum_j m_{ij} n_{ij}$ and therefore $\mathrm{tr}({}^t M.N) = \sum_{i,j=1,\cdots,d}\sum m_{ij} n_{ij}$.
3. It follows from the fact that $Diag(MN) = Diag(NM)$.
4. It follows from the prviou point $\mathrm{tr}(CMC^{-1}) = \mathrm{tr}(C^{-1}CM) = \mathrm{tr}(M)$.
5. From exercise 7 we can say that M and N are equivalent because both have rank 2 they are not similar since they have different traces.

Exercice 7

1. Comme on sait que les E_{ij} sont lineairement independant on a que :

$$\dim D_d = |\{E_{ii} : i = 1, .., d\}| = d$$

car c'est une famille generatrice et libre donc une base. Aussi

$$\dim T_d = |\{E_{ij} : i \leq j\}| = \frac{d(d-1)}{2}$$

car c'est une famille generatrice et libre donc une base. Le premier sev est le sev de matrices diagonales et deuxieme celui de matrices triangulaires superieures.

2. Il suffit de montrer que ces sevs sont stables par multiplication : Soient $M, N \in D_d$. Alors

$$(MN)_{ij} = \sum_{k=1}^{d} m_{ik} n_{kj} == \begin{cases} 0 & i \neq j \\ m_{ii} n_{ii} & sinon \end{cases}$$

Donc $MN \in D_d$. Si $M, N \in T_d$ on a

$$(MN)_{ij} = \sum_{k=1}^{d} m_{ik} n_{kj} = \sum_{i \leq k \leq j}^{d} m_{ik} n_{kj} = \begin{cases} 0 & i > j \\ \sum_{i \leq k \leq j}^{d} m_{ii} n_{ii} & \end{cases}$$

et on conclut

3. D_d est commutatif : On a

$$(MN)_{ii} = \sum_{k=1}^{d} m_{ik} n_{ki} = m_{ii} n_{ii} = n_{ii} m_{ii} = (NM)_{ij}$$

Mais T_d ne l'est pas : Si $M = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ et $N = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$ on a $MN \neq NM$

Exercice 8

On rappelle que pour $i, j \in 1, n$, on a $E_{i,j}E_{k,l} = \delta_{j,k}E_{i,l}$. En particulier on peut verifier que

$T_{i,j}^2 = (\mathrm{Id} - E_{i,i} - E_{j,j} + E_{i,j} + E_{j,i})^2$
$T_{i,j}^2 = \mathrm{Id} - E_{i,i} - E_{j,j} + E_{i,j} + E_{j,i} - E_{i,i} + E_{i,i} + \delta_{j,i}E_{j,j} - \delta_{i,j}E_{i,i} - E_{j,i} - E_{j,j}$
$+ \delta_{i,j}E_{i,i} + E_{j,j} - E_{i,j} - \delta_{i,j}E_{j,j} + E_{i,j} - E_{i,j} - \delta_{i,j}E_{j,j} + \delta_{i,j}E_{j,j} + E_{j,j} + E_{j,i} - \delta_{i,j}E_{j,j}$
$- E_{j,i} + E_{i,i} + \delta_{j,i}E_{j,j}$
$T_{i,j}^2 = \mathrm{Id}$

$$\begin{aligned}
D_{i,\lambda}D_{i,1/\lambda} &= (\mathrm{Id} + (\lambda - 1)E_{i,i})\Big(\mathrm{Id} + \Big(\frac{1}{\lambda} - 1\Big)E_{i,i}\Big) \\
D_{i,\lambda}D_{i,1/\lambda} &= (\mathrm{Id} + (\lambda - 1)E_{i,i})\Big(\mathrm{Id} + \Big(\frac{1-\lambda}{\lambda}\Big)E_{i,i}\Big) \\
D_{i,\lambda}D_{i,1/\lambda} &= \mathrm{Id} + (\lambda - 1)E_{i,i} + \Big(\frac{1-\lambda}{\lambda}\Big)E_{i,i} - \Big(\frac{(1-\lambda)^2}{\lambda}\Big)E_{i,i} \\
D_{i,\lambda}D_{i,1/\lambda} &= \mathrm{Id} + \Big(\frac{\lambda^2 - 2\lambda + 1 - (1-\lambda)^2}{\lambda}\Big)E_{i,i} \\
D_{i,\lambda}D_{i,1/\lambda} &= \mathrm{Id}
\end{aligned}$$

On pretera attention au fait que $Cl_{i,i,-1}$ pour $i \in 1, n$ n'est pas une transformation elementaire. De surcroit, on supposera $i \neq j$ autrement on est reduit au cas tout juste traite.

$$\begin{aligned}
Cl_{i,j,\mu}Cl_{i,j,-\mu} &= (\mathrm{Id} + \mu E_{i,j})(\mathrm{Id} - \mu E_{i,j}) \\
Cl_{i,j,\mu}Cl_{i,j,-\mu} &= \mathrm{Id} + \mu E_{i,j} - \mu E_{i,j}) \\
Cl_{i,j,\mu}Cl_{i,j,-\mu} &= \mathrm{Id}
\end{aligned}$$

Exercice 9

Si $car(K) = 2$ kla matrice sera

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

On observe que la premiere et la derniere ligne sont les memes. Donc en multipliant a gauche par $Cl_{13,-1}$ on a

$$Cl_{31,1}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix}=\begin{pmatrix}1&0&1&0\\0&1&0&1\\0&0&0&0\end{pmatrix}$$

Si $car(K)\neq 2$ on effectue les operations suivantes

$$Cl_{21,-2}Cl_{31,-3}\begin{pmatrix}1&2&3&4\\2&3&4&1\\3&4&1&2\end{pmatrix}=\begin{pmatrix}1&2&3&0\\0&-1&-2&-7\\0&-2&-8&-10\end{pmatrix}$$

$$Cl_{13,1}D_{2,-1}\begin{pmatrix}1&2&3&0\\0&-1&-2&-7\\0&-2&-8&-10\end{pmatrix}=\begin{pmatrix}1&0&-5&-6\\0&1&2&7\\0&-2&-8&-10\end{pmatrix}$$

$$Cl_{32,2}\begin{pmatrix}1&0&-5&-6\\0&1&2&7\\0&-2&-8&-10\end{pmatrix}=\begin{pmatrix}1&0&-5&-6\\0&1&2&7\\0&0&-4&4\end{pmatrix}$$

$$D_{2,-1/4}\begin{pmatrix}1&0&-5&-6\\0&1&2&7\\0&0&-4&4\end{pmatrix}=\begin{pmatrix}1&0&-5&-6\\0&1&2&7\\0&0&1&-1\end{pmatrix}$$

$$Cl_{13,5}Cl_{23,-2}\begin{pmatrix}1&0&-5&-6\\0&1&2&7\\0&0&1&-1\end{pmatrix}=\begin{pmatrix}1&0&0&-11\\0&1&0&9\\0&0&1&-1\end{pmatrix}$$

et on trouve donc que

$$D_{2,-1/4}Cl_{32,2}Cl_{13,1}D_{2,-1}Cl_{21,-2}Cl_{31,-3}M=\begin{pmatrix}1&0&0&-11\\0&1&0&9\\0&0&1&-1\end{pmatrix}$$