Introduction to Differentiable Manifolds
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Solutions Series 10 - Manifolds with boundary, line integrals and Tensors
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Manifolds with boundary

Exercise 10.1. Let M be a smooth n-dimensional manifold with boundary, prove that T_pM is an $n-$dimensional real vector space:

(a) First prove that $T_a\mathbb{H}^n \xrightarrow{d\iota} T_a\mathbb{R}^n$ is an isomorphism (using the fact that if f is a smooth function on $\mathbb{H}^n$ then there exists an extension $\widetilde{f}$ to a smooth function on all $\mathbb{R}^n$; look back at Exercise sheet 2 and 3)

(b) As we did in the case of smooth manifolds without boundary, prove that $T_pU \cong T_pM$ for each open U again using the Extension Lemma and then use smooth charts (Here, once again, remember what a smooth chart means in the case of a manifold with boundary)

Solution. See Proposition's 3.11 and 3.12 of Lee's "Introduction to Smooth Manifolds". □

Line integrals

Exercise 10.2. Let $M = \mathbb{R}^2 \setminus 0$. Consider the 1-form

$$\omega = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2}.$$

Let $\gamma : [0, 2\pi] \to M$ be the smooth curve defined by $t \mapsto (\cos t, \sin t)$.

(a) Compute the integral of ω along γ.

Solution. We compute first the pullback $\gamma^*\omega$, i.e. the 1-form on $[0, 2\pi]$ that assigns to each point $t \in [0, 2\pi]$ the covector

$$(\gamma^*\omega)|_t = \omega|_{\gamma(t)} \circ D\gamma|_t.$$

We express this covector as $\gamma^*\omega|_t = g(t)\ \mathrm{d}\tau$, where $\tau = \mathrm{id}_{\mathbb{R}}$ is the standard coordinate on $\mathbb{R}$, and $g : [0, 2\pi] \to \mathbb{R}$ is a function that we obtain by applying the covector $\gamma^*\omega|_t$ to the standard vector $\frac{\partial}{\partial\tau}\big|_t \in T_t\mathbb{R}$, that is,

$$\begin{aligned} g(t) &= (\gamma^*\omega)|_t \left(\frac{\partial}{\partial\tau}\bigg|_t \right) \\ &= \omega|_{\gamma(t)} \left(D\gamma|_t\, \frac{\partial}{\partial\tau}\bigg|_t \right) \\ &= \omega|_{\gamma(t)}(\gamma'(t)) \\ &= (\cos t\ \mathrm{d}y - y\ \mathrm{d}x)|_{\gamma(t)}\ \left(-\sin t\, \frac{\partial}{\partial x} + \cos t\, \frac{\partial}{\partial y} \right)\bigg|_{\gamma(t)} \\ &= (\cos t)^2 + (\sin t)^2 = 1. \end{aligned}$$

This means that $\gamma^*\omega = \mathrm{d}\tau$.

Now we can compute the integral

$$\int_\gamma \omega = \int_{[0,2\pi]} \gamma^*\omega = \int_{[0,2\pi]} h(t)\ \mathrm{d}t = \int_{[0,2\pi]} 1\ \mathrm{d}t = 2\pi$$

□

(b) Prove that omega is not exact, i.e. is not of the form $\mathrm{d}h$ for $h \in C^\infty(M)$.

Solution. The integral of an exact 1-form along a closed path is zero by Stokes' theorem. Since γ is a closed path and the integral of ω along γ is not zero, we conclude that ω is not exact. □

Exercise 10.3 (to hand in). Consider the following 1-form on $M = \mathbb{R}^3$:

$$\omega = \frac{-4z\,\mathrm{d}x}{(x^2+1)^2} + \frac{2y\,\mathrm{d}y}{y^2+1} + \frac{2x\,\mathrm{d}z}{x^2+1}$$

(a) Set up and compute the line integral of ω along the line going from $(0,0,0)$ to $(1,1,1)$

(b) Consider the smooth map $\Psi : W \to \mathbb{R}^3$ given by $(r,\varphi,\theta) \in W := \mathbb{R}^+ \times (0,2\pi) \times (0,\pi)$:

$$\Psi(r,\varphi,\theta) = (r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta) \in \mathbb{R}^3.$$

Compute $\Psi^*\omega$.

Exercise 10.4. On the plane $\mathbb{R}^2$ with the standard coordinates (x,y) consider the 1-form $\theta = x\,\mathrm{d}y$. Compute the integral of θ along each side of the square $[1,2]\times[3,4]$, with each of the two orientations. (There are 8 numbers to compute.)

Solution. The four integrals along the horizontal sides are zero because $\mathrm{d}y \equiv 0$ on any horizontal line.

Along a vertical line given by an equation $x = c$, with $c \in \mathbb{R}$ a constant, the vector field θ coincides with the 1-form $c\,\mathrm{d}y \in \Omega^1(\mathbb{R}^2)$, which is the differential of the function $h_c(x,y) = cy$. Therefore the integral of θ along a segment of such a vertical line is equal to the variation of the function h_c along this segment.

Along the segment $\{2\}\times[3,4]$ we have $c = 2$, thus the integral of θ is 2 if we go upwards and -2 if we go downwards. Similarly, along the segment $\{1\}\times[3,4]$ we have $c = 1$, thus the integral of θ is 1 if we go upwards and -1 if we go downwards. □

Tensors

Exercise 10.5. Let $\mathcal{B} = (E_i)_i$ and $\widetilde{\mathcal{B}} = (\widetilde{E}_j)_j$ be two bases of a vector space $V \simeq \mathbb{R}^n$, and let $\mathcal{B}^* = (\varepsilon^i)_i$ and $\widetilde{\mathcal{B}}^* = (\widetilde{\varepsilon}^j)_j$ be the respective dual bases. Note that a tensor $T \in \operatorname{Ten}^k V$ can be written as

$$T = \sum_{i_0,\dots,i_{k-1}} T_{i_0,\dots,i_{k-1}}\,\varepsilon^{i_0}\otimes\cdots\otimes\varepsilon^{i_{k-1}} \quad \text{or as} \quad T = \sum_{j_0,\dots,j_{k-1}} \widetilde{T}_{j_0,\dots,j_{k-1}}\,\widetilde{\varepsilon}^{j_0}\otimes\cdots\otimes\widetilde{\varepsilon}^{j_{k-1}}.$$

Find the transformation law that expresses the coefficients $\widetilde{T}_{j_0,\dots,j_{k-1}}$ in terms of the coefficients $T_{i_0,\dots,i_{k-1}}$.

Solution. There exists an invertible $n\times n$ matrix $(a^i_j)_{i,j\in\underline{n}}$ such that $\widetilde{E}_j = \sum_i a^i_j E_i$. For any k-index $J = (j_0,\dots,j_{k-1}) \in \underline{n}^k$ we have

$$\begin{aligned}
\widetilde{T}_J = T(\widetilde{E}_J) = T(\widetilde{E}_{j_0},\dots,\widetilde{E}_{j_{k-1}}) &= T\left(\sum_{i_0\in\underline{n}} a^{i_0}_{j_0}\,E_{i_0},\dots,\sum_{i_{k-1}\in\underline{n}} a^{i_{k-1}}_{j_{k-1}}\,E_{i_{k-1}}\right)\\
&= \sum_{I\in\underline{n}^k} a^{i_0}_{j_0}\cdots a^{i_{k-1}}_{j_{k-1}} T(E_{i_0},\dots,E_{i_{k-1}})\\
&= \sum_{I\in\underline{n}^k} a^{i_0}_{j_0}\cdots a^{i_{k-1}}_{j_{k-1}}\,T_I.
\end{aligned}$$

□

Exercise 10.6 (Alternating covariant tensors)**.** Let V be a finite-dimensional real vector space. Let $T \in \operatorname{Ten}^k V$. Suppose that with respect to some basis ε^i of V^*

$$T = \sum_{1\le i_1,\dots,i_k<n} T_{i_1\cdots i_k}\varepsilon^{i_1}\otimes\cdots\otimes\varepsilon^{i_k}.$$

Show that T is alternating iff:

(a) for all $\sigma \in S_k$: $T_{i_{\sigma(1)}\cdots i_{\sigma(k)}} = \operatorname{sgn}(\sigma)\, T_{i_1\cdots i_k}$.

Solution. Suppose that T is alternating, then for all $\sigma \in S_n$ we have

$$T_{i_{\sigma(1)},\dots,i_{\sigma(k)}} = T(E_{i_{\sigma(1)}},\dots,E_{i_{\sigma(k)}}) = \operatorname{sgn}(\sigma)\, T(E_{i_1},\dots,E_{i_k}) = \operatorname{sgn}(\sigma)\, T_{i_1,\dots,i_k}$$

Conversely if $T_{i_{\sigma(1)},\dots,i_{\sigma(k)}} = \operatorname{sgn}\sigma\, T_{i_1,\dots,i_k}$ for all $\sigma \in S_n$, then in particular

$$T_{\alpha_1,\dots,\alpha_i,\dots,\alpha_j,\dots,\alpha_k} = -T_{\alpha_1,\dots,\alpha_j,\dots,\alpha_i,\dots,\alpha_k}$$

Then for any $X_1,\dots,X_k \in V$ the multi-linearity of T yields (we use the summation convention where repeated indices are summed over)

$$\begin{aligned}
&T(X_1,\dots,X_i,\dots,X_j,\dots,X_k)\\
&= T(X_1^{\alpha_1}E_{\alpha_1},\dots,X_i^{\alpha_i}E_{\alpha_i},\dots,X_j^{\alpha_j}E_{\alpha_j},\dots,X_k^{\alpha_k}E_{\alpha_k})\\
&= X_1^{\alpha_1}\dots X_k^{\alpha_k}T(E_{\alpha_1},\dots,E_{\alpha_i},\dots,E_{\alpha_j},\dots,E_{\alpha_k})\\
&= -X_1^{\alpha_1}\dots X_k^{\alpha_k}T(E_{\alpha_1},\dots,E_{\alpha_j},\dots,E_{\alpha_i},\dots,E_{\alpha_k})\\
&= -T(X_1,\dots,X_j,\dots,X_i,\dots,X_k)
\end{aligned}$$

Hence T is alternating. □

(b) $T(\dots,X_s,\dots,X_t,\dots) = -T(\dots,X_s,\dots,X_t,\dots)$

Solution. Both sides are multilinear in the ω^i and so the result follows from the one for the basis covectors $\omega^i = \varepsilon^{\ell_i}$, which we have seen in the lecture.

Nevertheless, let us carry out the argument in detail. In the lecture we saw

$$\varepsilon^{\ell_1} \wedge \cdots \wedge \varepsilon^{\ell_k}(X_1,\dots,X_k) = \det(\varepsilon^{\ell_r}(X_j))^j_r$$

(on the right hand side we have the determinant of a $k \times k$ matrix ($r, j = 1,\dots k$; think of j as the column index and r as the row index).

Now for arbitrary covectors $\omega^r = \sum_{\ell=1}^n \omega^r_\ell \varepsilon^\ell$ we have (for the first equality we use the multilinearity of the wedge product)

$$\begin{aligned}
\omega^1 \wedge \cdots \wedge \omega^k(X_1,\dots,X_k) &= \sum_{\ell_1=1}^n \cdots \sum_{\ell_k=1}^n \omega^1_{\ell_1}\cdots\omega^k_{\ell_k}\varepsilon^{\ell_1}\wedge\cdots\wedge\varepsilon^{\ell_k}(X_1,\dots,X_k)\\
&= \sum_{\ell_1=1}^n \cdots \sum_{\ell_k=1}^n \omega^1_{\ell_1}\cdots\omega^k_{\ell_k}\det(\varepsilon^{\ell_r}(X_j))^j_r\\
&= \sum_{\ell_1=1}^n \cdots \sum_{\ell_k=1}^n \omega^1_{\ell_1}\cdots\omega^k_{\ell_k}\det\begin{pmatrix}\varepsilon^{\ell_1}(X_1) & \cdots & \varepsilon^{\ell_1}(X_k)\\ \vdots & & \vdots\\ \varepsilon^{\ell_k}(X_1) & \cdots & \varepsilon^{\ell_k}(X_k)\end{pmatrix}\\
&= \det(\omega^r(X_j))^i_r
\end{aligned}$$

where in the last step we multiplied the r-th line by $\omega^r_{\ell_r}$ and replaced $\sum_{\ell_r=1}^n \omega^r_{\ell_r}\varepsilon^{\ell_r} = \omega^r$.

□