

## Astrophysics III: Stellar and galactic dynamics

### Exercises

**Problem 1:** Derive the *linearised collisionless Boltzmann equation* (5.11 in Binney & Tremaine 1987) from the course:

$$\frac{\partial f_1}{\partial t} + [f_1, H_0] + [f_0, \Phi_1] = 0 \quad ; \quad \nabla^2 \Phi_{s1} = 4\pi G \int d^3\mathbf{v} f_1 \quad (1)$$

**Definitions and hints:** The Poisson bracket is defined as

$$[A, B] \equiv \frac{\partial A}{\partial \mathbf{q}} \cdot \frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}} \cdot \frac{\partial B}{\partial \mathbf{q}} \quad (2)$$

where  $A$  and  $B$  are any scalar functions of the phase-space coordinates.

We start from the collisionless Boltzmann equation and from Poisson's equation

$$\frac{\partial f}{\partial t} + [f, H] = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \quad (3)$$

$$\nabla^2 \Phi_s(\mathbf{x}, t) = 4\pi G \int d^3\mathbf{v} f(\mathbf{x}, \mathbf{v}, t) \quad , \quad (4)$$

where  $\Phi(\mathbf{x}, t)$  is the total potential,  $H = \frac{1}{2}v^2 + \Phi(\mathbf{x}, t)$  is the Hamiltonian, and  $\Phi_s(\mathbf{x}, t)$  is the gravitational potential of the stellar system, which may differ from the total potential  $\Phi(\mathbf{x}, t)$  if there is an external perturbing potential  $\Phi_e(\mathbf{x}, t)$ . An isolated stellar system, as far as it is in equilibrium, is described by time-independent DF  $f_0(\mathbf{x}, \mathbf{v})$  and potential  $\Phi_0(\mathbf{x})$  that are solutions of 3 and 4

$$[f_0, H_0] = 0 \quad ; \quad \nabla^2 \Phi_0 = 4\pi G \int d^3\mathbf{v} f_0 \quad \text{with} \quad H_0 = \frac{1}{2}v^2 + \Phi_0(\mathbf{x}) \quad (5)$$

Now we assume that the equilibrium system is subjected to a weak external potential  $\epsilon\Phi_e(\mathbf{x}, t)$ , where  $|\nabla\Phi_e|$  is of order  $|\nabla\Phi_0|$  and  $\epsilon \ll 1$ . In response to this disturbance, the DF of the stellar system and the potential arising from its stars become

$$f(\mathbf{x}, \mathbf{v}, t) = f_0(\mathbf{x}, \mathbf{v}) + \epsilon f_1(\mathbf{x}, \mathbf{v}, t) \quad ; \quad \Phi_s(\mathbf{x}, t) = \Phi_0(\mathbf{x}) + \epsilon\Phi_{s1}(\mathbf{x}, t) \quad (6)$$

and the total potential becomes

$$\Phi(\mathbf{x}, t) = \Phi_0(\mathbf{x}, t) + \epsilon\Phi_1(\mathbf{x}, t) \quad \text{with} \quad \Phi_1(\mathbf{x}, t) = \Phi_{s1}(\mathbf{x}, t) + \Phi_e(\mathbf{x}, t) \quad (7)$$

**Problem 2:** Derive equations (5.23) to (5.26) in Binney & Tremaine 1987 for linearized fluid systems in the course:

$$\frac{\partial \rho_{s1}}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}_1) + \nabla \cdot (\rho_{s1} \mathbf{v}_0) = 0 \quad (8)$$

$$\frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_0 \cdot \nabla) \mathbf{v}_1 + (\mathbf{v}_1 \cdot \nabla) \mathbf{v}_0 = -\nabla(h_1 + \Phi_{s1} + \Phi_e) \quad (9)$$

$$\nabla^2 \Phi_{s1} = 4\pi G \rho_{s1} \quad (10)$$

$$h_1 = \frac{p_1}{\rho_0} = \left( \frac{dp}{d\rho} \right)_{\rho_0} \frac{\rho_{s1}}{\rho_0} = v_s^2 \frac{\rho_{s1}}{\rho_0} \quad (11)$$

knowing that the system is characterized by a density  $\rho_s(\mathbf{x}, t)$ , a pressure  $p(\mathbf{x}, t)$ , a velocity  $\mathbf{v}(\mathbf{x}, t)$  and a potential  $\Phi(\mathbf{x}, t)$ , quantities that are linked by the continuity, Euler's and Poisson's equations:

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{v}) = 0, \quad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho_s} \nabla p - \nabla \Phi, \quad \nabla^2 \Phi_s = 4\pi G \rho_s. \quad (12)$$

One has  $\Phi = \Phi_s + \epsilon \Phi_e$  and the equation of state is assumed barotropic:  $p(\mathbf{x}, t) = p[\rho_s(\mathbf{x}, t)]$ . Thus, introducing the specific enthalpy  $h$ , Euler's equation becomes

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla(h + \Phi) \quad \text{with} \quad h(\rho_s) \equiv \int_0^{\rho_s} \frac{dp(\rho)}{\rho}. \quad (13)$$

We further introduce the sound velocity  $v_s^2(\mathbf{x}) \equiv \left[ \frac{dp(\rho)}{d\rho} \right]_{\rho_0(\mathbf{x})}$ . The response of the fluid to a weak external potential  $\epsilon \Phi_e(\mathbf{x}, t)$  is

$$\begin{aligned} \rho_s(\mathbf{x}, t) &= \rho_0(\mathbf{x}) + \epsilon \rho_{s1}(\mathbf{x}, t) & ; & \quad h(\mathbf{x}, t) = h_0(\mathbf{x}) + \epsilon h_1(\mathbf{x}, t) \\ \mathbf{v}(\mathbf{x}, t) &= \mathbf{v}_0(\mathbf{x}) + \epsilon \mathbf{v}_1(\mathbf{x}, t) & ; & \quad \Phi(\mathbf{x}, t) = \Phi_0(\mathbf{x}) + \epsilon \Phi_1(\mathbf{x}, t) \end{aligned} \quad (14)$$

where  $\Phi_1 = \Phi_{s1} + \Phi_e$  is the total perturbation in the potential, the sum of the external potential  $\Phi_e$  and of the  $\Phi_{s1}$  potential arising from the density perturbation  $\rho_{s1}$ .

**Problem 3:**

Show that the density of a gaseous sphere, which had initial density  $\rho_0$ , when compressed by a factor  $\epsilon$  (i.e.,  $r_1 = (1 - \epsilon)r$ ) increases in proportion to  $\epsilon \rho_0$ . Similarly, show that its pressure increases proportionally to  $v_s^2 \epsilon \rho_0$ , where  $v_s$  is the speed of sound.

Then, using order-of-magnitude estimates, find expressions for the force changes of the pressure and gravity introduced by the contraction, and find the instability criterion expressed by  $r$ .