Solution 11
Quantum Information Processing

Exercise 1 *W states, reduced density matrix, entropy*

(a)

$$
\begin{aligned}
|W_\theta\rangle\langle W_\theta| &= \frac{\cos^2\theta}{2}|100\rangle\langle 100| + \frac{\cos^2\theta}{2}|010\rangle\langle 010| + \sin^2\theta\,|001\rangle\langle 001| \\
&+ \frac{\cos^2\theta}{2}\big(|100\rangle\langle 010| + |010\rangle\langle 100|\big) + \frac{\cos\theta\sin\theta}{\sqrt{2}}\big(|100\rangle\langle 001| + |001\rangle\langle 100|\big) \\
&+ \frac{\cos\theta\sin\theta}{\sqrt{2}}\big(|010\rangle\langle 001| + |001\rangle\langle 010|\big)
\end{aligned}
$$

So we find (using cyclicity of trace and inner product for AB system)

$$\rho_C = \mathrm{Tr}_{AB}[|W_\theta\rangle\langle W_\theta|] = \cos^2\theta\,|0\rangle\langle 0| + \sin\theta^2\,|1\rangle\langle 1| \tag{1}$$

And (using cyclicity of trace and inner product for C system)

$$
\begin{aligned}
\rho_{AB} = \mathrm{Tr}_C[|W\rangle\langle W|] &= (\cos\theta)^2\,(|10\rangle\langle 10| + |10\rangle\langle 01|) &(2)\\
&+ (\cos\theta)^2\,(|01\rangle\langle 10| + |01\rangle\langle 01|) &(3)\\
&+ (\sin\theta)^2\,|00\rangle\langle 00| &(4)\\
&= (\cos\theta)^2\,|\beta_{01}\rangle\langle\beta_{01}| + (\sin\theta)^2\,|00\rangle\langle 00| &(5)
\end{aligned}
$$

(b) ρ_{AB} is a matrix of size 4×4 while ρ_C is of size 2×2. They are both rank 2 with non-zero eigenvalues $(\cos\theta)^2$ and $(\sin\theta)^2$ (notice that $|\beta_{01}\rangle$ is orthonormal with $|00\rangle$). Note that the matrix ρ_{AB} has two extra zero eigenvalues.

(c) In both cases, the Von Neumann entropy is:

$$S = -(\cos^2\theta)\log\big(\cos^2\theta\big) - (\sin^2\theta)\log\big(\sin^2\theta\big) \tag{6}$$

(d) Here we can find the condition $(\sin\theta)^2 < \frac{\sqrt{2}-1}{\sqrt{2}+1}$ without much calculation in the following way. By linearity and cyclicity

$$\mathrm{Tr}\mathcal{B}\rho_{AB} = (\sin\theta)^2\,\langle 00|\,\mathcal{B}\,|00\rangle + (\cos\theta)^2\,\langle\beta_{01}|\,\mathcal{B}\,|\beta_{01}\rangle$$

Let us take the angles that maximize the term $\langle\beta_{01}|\,\mathcal{B}\,|\beta_{01}\rangle$ and make it equal to $2\sqrt{2}$. For the other term since it is a product state we must certainly have $\langle 00|\,\mathcal{B}\,|00\rangle \geq -2$. Thus we get for these angles:

$$\mathrm{Tr}\mathcal{B}\rho_{AB} \geq -2(\sin\theta)^2 + 2\sqrt{2}(\cos\theta)^2$$

To check violation of the Bell inequality we impose $-2(\sin\theta)^2 + 2\sqrt{2}(\cos\theta)^2 > 2$ which gives the condition $(\sin\theta)^2 < \frac{\sqrt{2}-1}{\sqrt{2}+1}$.

The computation for al angles of the average of the Bell operator is done as follows.

$$\langle 00| A \otimes B |00\rangle = (\cos(\alpha)^2 - \sin(\alpha)^2)(\cos(\beta)^2 - \sin(\beta)^2) = \cos(2\alpha)\cos(2\beta) \tag{7}$$

On the other hand, notice: $|\beta_{01}\rangle = (X \otimes I)|\beta_{00}\rangle$ So in fact, with $\tilde{A} = XAX$ we have:

$$\langle \beta_{01}| A \otimes B |\beta_{01}\rangle = \langle \beta_{00}| \tilde{A} \otimes B |\beta_{00}\rangle \tag{8}$$

Now it can be checked that with $\tilde{\alpha} = \frac{\pi}{2} - \alpha$ we have:

$$\tilde{A} = XAX = |\tilde{\alpha}\rangle\langle\tilde{\alpha}| - |\tilde{\alpha}^{\perp}\rangle\langle\tilde{\alpha}^{\perp}| \tag{9}$$

Hence using the formula from the course:

$$\langle \beta_{01}| A \otimes B |\beta_{01}\rangle = \cos(2(\tilde{\alpha} - \beta)) = \cos(\pi - 2(\alpha + \beta)) = -\cos(2(\alpha + \beta)) \tag{10}$$

Putting things together gives a general expression for $\mathrm{Tr}\mathcal{B}\rho_{AB}$ in terms of θ, α, β, α', β' which however is not easily optimized (if one would like to find angles that maximize it for given θ).

For the last question: note that for a density matrix of the form $\rho_A \otimes \rho_B$ the locality assumption is true i.e. $p(a,b|\alpha,\beta) = p(a|\alpha)p(b|\beta)$. Indeed if say A and B choose the α, β measurement basis the probablity distributions are (by the measurement principle for mixed states)

$$p(a,b|\alpha,\beta) = \langle \alpha,\beta| \rho_A \otimes \rho_B |\alpha,\beta\rangle$$

and

$$p(a|\alpha) = \langle\alpha|\rho_A|\alpha\rangle, \quad p(a|\alpha) = \langle\alpha|\rho_A|\alpha\rangle$$

Thus by the general theory $|\mathrm{Tr}\mathcal{B}\rho_A \otimes \rho_B| \leq 2$. Hence this is also true for any convex combination $\sum_i p_i \rho_A^i \otimes \rho_B^i$

Exercise 2 *Dynamics of one-qubit density matrix*

(a) From Homework 5 we can write down with $\alpha_t^2 + \beta_t^2 = 1$ and $n_x^2 + n_z^2 = 1$:

$$U_t = \alpha I + \beta\left(n_x\sigma_x + n_z\sigma_z\right) \tag{11}$$

Now we can compute:

$$\rho_t = U_t \rho_0 U_t^{\dagger} \tag{12}$$

After a (long) calculation, one finds:

$$a_x(t) = a_x(0)\left(\alpha^2 + \beta^2 n_x^2 - \beta^2 n_z^2\right) - 2a_y(0)\alpha\beta n_z + 2a_z(0)\beta^2 n_x n_z \tag{13}$$
$$a_y(t) = 2a_x(0)\alpha\beta n_z + a_y(0)\left(\alpha^2 - \beta^2 n_x^2 - \beta^2 n_z^2\right) - 2a_z(0)\alpha\beta n_x \tag{14}$$
$$a_z(t) = 2a_x(0)\beta^2 n_x n_z + 2a_y(0)\alpha\beta n_x + a_z(0)\left(\alpha^2 - \beta^2 n_x^2 + \beta^2 n_z^2\right) \tag{15}$$

(b) One can check this after a long calculation using the previous formulas

(c) It suffices to notice that $1 - \|a_t\|^2 = \det(\rho_t) = \det(\rho) = 1 - \|a\|^2$