Stability of collisionless systems

2nd part

Outlines

Linear response theory

- in fluid systems
- in stellar systems

The Jeans instability

- in fluid systems
- in stellar systems

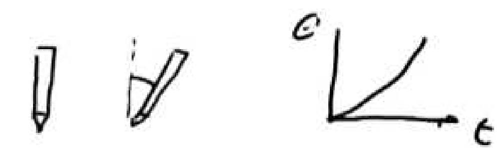
Our goal Study the stability of systems at equilibrium

Method : perturbation theory

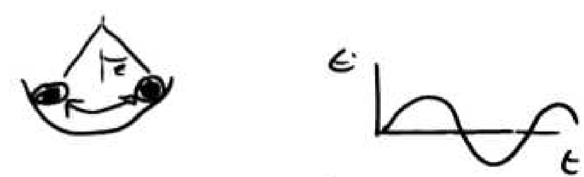
perturbation → **response**

Types of responses

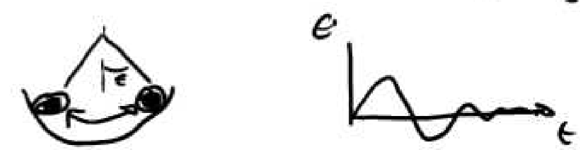
- Exponential growth of the perturbation



- Oscillation of the perturbation



- Die of the perturbation



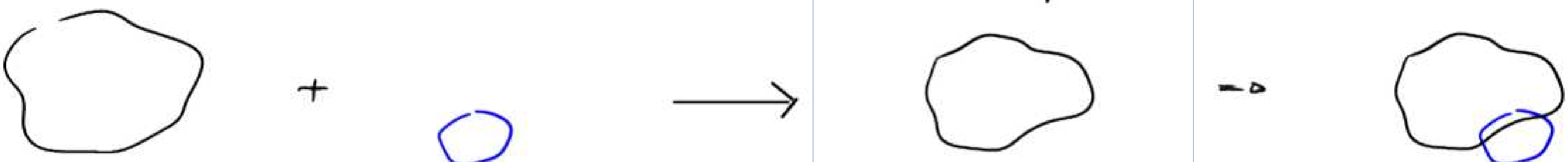
Stability of collisionless systems

Linear response theory

Linear response theory

Study the perturbation of system at the equilibrium

Perturbation: external gravitational field $-\varepsilon \vec{\nabla}\phi_e$: $|\vec{\nabla}\phi_e| \sim |\vec{\nabla}\phi_0|$



stellar/system at equilibrium	external perturbation	stellar/fluid responses linearly	total stellar/fluid linear response
$\rho_{s_0}(x)$	$\varepsilon\,\rho_e(x)$	$\rho_s = \rho_{s_0} + \varepsilon \rho_{s_1}$	$\rho_{tot} = \rho_{s_0} + \varepsilon\rho_{s_1} + \varepsilon\rho_e$ $= \rho_{s_0} + \varepsilon\,\rho_1$
$\phi_{s_0}(x)$	$\varepsilon\,\phi_e(x)$	$\phi_s = \phi_{s_0} + \varepsilon\,\phi_{s_1}$	$\phi_{tot} = \phi_{s_0} + \varepsilon\phi_{s_1} + \varepsilon\phi_e$ $= \phi_{s_0} + \varepsilon\,\phi_1$
$\nabla^2\phi_{s_0} = 4\pi G \rho_{s_0}$	$\nabla^2\phi_e = 4\pi G\rho_e$	$\nabla^2\phi_s = 4\pi G\,\rho_s$ $\nabla^2\phi_{s_1} = 4\pi G\,\rho_{s_1}$	$\nabla^2\phi_1 = 4\pi G\,\rho_1$ $\nabla^2\phi_{tot} = 4\pi G\,\rho_{tot}$

Conventions: $\rho_{s_0} = \rho_0$ $\quad \phi_{s_0} = \phi_0$ $\quad \rho_{tot} = \rho$ $\quad \phi_{tot} = \phi$

Linearized equations for a self-gravitating system

Self-gravitating fluid

$\rho(\vec{x},t) \quad \phi(\vec{x},t) \quad p(\vec{x},t) \quad \vec{v}(\vec{x},t)$

6 Equ.
6 Unkn.

① Continuity Equation

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0}$$

② Euler Equation

$$\boxed{\frac{\partial}{\partial t}\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\rho} - \vec{\nabla}\phi}$$

③ The Poisson Equation

$$\boxed{\nabla^2\phi = 4\pi G\rho}$$

④ Equation of state (barotropic)

$$\boxed{p(\vec{x},t) = p(\rho(x,t))}$$

Self-gravitating stellar system

$f(\vec{x},\vec{v},t)\,,\quad \rho(\vec{x},t)\,,\quad \phi(\vec{x},t)$

3 Equ.
3 Unkn.

① The collisionless Boltzmann Equation

$$\boxed{\frac{\partial f}{\partial t} + \vec{v}\,\frac{\partial f}{\partial \vec{x}} - \frac{\partial \phi}{\partial \vec{x}}\,\frac{\partial f}{\partial \vec{v}} = 0} \quad = \quad \frac{\partial f}{\partial t} + [f, H] = 0$$

② The Poisson Equation

$$\boxed{\nabla^2\phi = 4\pi G\rho = 4\pi G\int d^3v\, f(\vec{x},\vec{v},t)}$$

Linearizing equations

Goal : describe the response of the system (without the perturbation)

$$\boxed{\rho_s(\bar{x},t) = \rho_{s_0}(\bar{x}) + \varepsilon\, \rho_{s_1}(\bar{x},t)}$$

$\phi_s \quad \hat{v}_s \quad p_s \quad \rho_s$

1. Equations for $\rho_s(\bar{x},t)$

2. Equations for $\rho_{s_0}(\bar{x})$ (system at equilibrium)

3. Keep only first order terms $(\varepsilon^2 \to 0)$

$\boxed{\text{linear differential equations for } \rho_{s_1}}$ $\quad \phi_{s_1},\ \hat{v}_{s_1},\ p_{s_1},\ \rho_{s_1}$

Linearized Equations for a self-gravitating fluid

Equations for a self-gravitating fluid (we follow only the system, without the perturbation)

system of density $\rho_s(\vec{x},t)$ $\phi_s(\vec{x},t)$ $\vec{v}_s(x,t)$ $p_s(\vec{x},t)$

① Continuity Equation

$$\boxed{\frac{\partial \rho_s}{\partial t} + \vec{\nabla}(\rho_s \vec{v}_s) = 0}$$

total potential $\phi = \phi_s + \varepsilon \phi_e$

② Euler Equation

$$\boxed{\frac{\partial}{\partial t}\vec{v}_s + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s = -\frac{\vec{\nabla} p_s}{\rho_s} - \vec{\nabla}\phi}$$

③ The Poisson Equation

$$\boxed{\nabla^2 \phi_s = 4\pi G \rho_s}$$

④ Equation of state

$$\boxed{p_s = p(\rho_s)}$$

<u>Definition</u> : $\boxed{\text{specific enthalpy}}$

$$h(\rho_s) = \int_0^{\rho_s} \frac{1}{\rho'} \frac{\partial P_s}{\partial \rho_s}(\rho')\, d\rho'$$

for a barotropic EOS $P = P(\rho)$

$$\boxed{\vec{\nabla} h_s = \frac{1}{\rho_s} \vec{\nabla} P_s}$$

Euler equation becomes

$$\boxed{\frac{\partial}{\partial t} \vec{v}_s + (\vec{v}_s \cdot \vec{\nabla}) \vec{v}_s = - \frac{\vec{\nabla} P_s}{\rho_s} - \vec{\nabla} \phi}$$

$$\boxed{\frac{\partial}{t} \vec{v}_s + (\vec{v}_s \cdot \vec{\nabla}) \vec{v}_s = - \vec{\nabla}(h_s + \phi)}$$

<u>Definition</u> : $\boxed{\text{sound speed}}$

$$\boxed{V_s^2 = \left.\frac{\partial P(\rho)}{\partial \rho}\right|_{\rho_{s0}}}$$

for the unperturbed system

More on enthalpy

$$\boxed{h := u + \frac{P}{\rho}}$$

$$\underbrace{u}_{\text{spec. internal energy}} \qquad \underbrace{\frac{P}{\rho}}_{\text{mechanical energy}}$$

with
$$du = T\,dS + \frac{P}{\rho^2}\,d\rho$$
$$dh = du + \frac{dP}{\rho} - \frac{P}{\rho^2}\,d\rho$$
$$dh = T\,dS + \frac{dP}{\rho}$$

For $S = \text{cte}$ and $P = P(\rho)$ (barotropic EOS)

$$dh = \frac{dP}{\rho} = \frac{1}{\rho}\frac{\partial P}{\partial \rho}\,d\rho$$

Thus

$$h(\rho) = \int_0^\rho \frac{1}{\rho'}\frac{\partial P}{\partial \rho}(\rho')\,d\rho'$$

Isolated fluid at equilibrium

$\rho_{s0}(\vec{x})$ $\vec{v}_{s0}(x)$ $h_{s0}(\vec{x})$ $\phi_{s0}(x)$

solutions of

① Continuity Equation

$$\boxed{\cancel{\frac{\partial \rho_{s0}}{\partial t}} + \vec{\nabla}(\rho_{s0}\vec{v}_{s0}) = 0}$$

② Euler Equation

$$\boxed{\cancel{\frac{\partial}{\partial t}}\vec{v}_{s0} + (\vec{v}_{s0}\cdot\vec{\nabla})\,\vec{v}_{s0} = -\vec{\nabla}(h_{s0}+\phi_{s0})}$$

③ The Poisson Equation

$$\boxed{\nabla^2\phi_{s0} = 4\pi G\rho_{s0}}$$

④ Equation of state

$$\boxed{h_{s0}(\rho_{s0}) = \int_0^{\rho_{s0}} \frac{1}{\rho'}\,\frac{\partial P_{s0}}{\partial \rho_{s0}}(\rho')\,d\rho'}$$

$$P_{s0} = P(\rho_{s0})$$

The response of the system to a weak perturbation $-\varepsilon \vec{\nabla} \phi_e$

$$\rho_{s0}(\vec{x}) \longrightarrow \rho_s(\vec{x},t) = \rho_{s0}(\vec{x}) + \varepsilon \rho_{s1}(\vec{x},t)$$

$$h_{s0}(\vec{x}) \longrightarrow h_s(\vec{x},t) = h_{s0}(\vec{x}) + \varepsilon h_{s1}(\vec{x},t)$$

$$\vec{v}_{s0}(\vec{x}) \longrightarrow \vec{v}_s(\vec{x},t) = \vec{v}_{s0}(\vec{x}) + \varepsilon \vec{v}_{s1}(\vec{x},t)$$

$$\phi_{s0}(x) \longrightarrow \phi_s(x,t) = \phi_{s0}(\vec{x}) + \varepsilon \phi_{s1}(\vec{x},t)$$

system only, without the perturbation

$$\phi_{s0}(x) \equiv \phi_{0,\mathrm{tot}}(x) \longrightarrow \phi(x,t) = \phi_{s0}(\vec{x}) + \varepsilon \phi_1(\vec{x},t)$$

total pot. include the perturbation

$$= \phi_{s0}(\vec{x}) + \varepsilon \phi_{s1}(\vec{x},t) + \underbrace{\varepsilon \phi_e(\vec{x},t)}_{\text{contrib. of the ext. perturb.}}$$

→ insert those equations into the equations for a self-gravitating fluid $\rho_s(\vec{x},t)$ $\phi_s(\vec{x},t)$ $\vec{v}_s(x,t)$ $h_s(\vec{x},t)$

→ use equations for the unperturbed fluid

→ keep only first order terms $(\varepsilon^2 \to 0)$

We get a set of linear differential equations

→ time evolution of the perturbation

① "Continuity Equation"

ρ_{s1} $\vec{v}_{s1}$ ϕ_{s1} h_{s1}

6 unknowns

6 equations

$$\frac{\partial}{\partial t} \rho_{s1} + \vec{\nabla}(\rho_{s0} \vec{v}_{s1}) + \vec{\nabla}(\rho_{s1} \vec{v}_{s0}) = 0$$

② "Euler Equation"

$$\frac{\partial}{\partial t} \vec{v}_{s1} + \vec{v}_{s0} \cdot (\vec{\nabla} \cdot \vec{v}_{s1}) + \vec{v}_{s1} (\vec{\nabla} \cdot \vec{v}_{s0}) = - \vec{\nabla} (h_{s1} + \phi_{s1} + \phi_e)$$

$$= - \frac{\vec{\nabla} P_{s1}}{\rho_{s0}} - \vec{\nabla} (\phi_{s1} + \phi_e)$$

③ "Poisson Equation"

$$\vec{\nabla} \phi_{s1} = 4 \pi G \rho_{s1}$$

④ "Equation of state"

$$h_{s1} = \left. \frac{d P(\rho)}{d\rho} \right|_{\rho_0} \frac{\rho_{s1}}{\rho_{s0}} = v_s^2 \frac{\rho_{s1}}{\rho_{s0}}$$

Linearized Equations for a self-gravitating stellar system

Equations for a self-gravitating steller system

System of DF $\quad f_s(\bar{x},\bar{v},t) \rightarrow \rho_s(\bar{x},t) \quad \phi_s(\bar{x},t)$

(without the perturbation)

① The collisionless Boltzmann Equation

total potential $\phi = \phi_s + \varepsilon \phi_e$

$$\boxed{\frac{\partial f_s}{\partial t} + \bar{v}\frac{\partial f_s}{\partial \bar{x}} - \frac{\partial \phi}{\partial \bar{x}}\frac{\partial f_s}{\partial \bar{v}} = 0} \quad = \quad \frac{\partial f_s}{\partial t} + [f_s, H] = 0$$

$$H = \frac{1}{2}\bar{v}^2 + \phi(\bar{x},t)$$

② The Poisson Equation

$$\boxed{\nabla^2 \phi_s = 4\pi G \rho_s = 4\pi G \int d^3v\, f_s(\bar{x},\bar{v},t)}$$

Isolated stellar system at equilibrium $\quad f_0(\vec{x},\vec{v}) \quad \rho_0(\vec{x}) \quad \phi_0(\vec{x})$

solutions of

① The collisionless Boltzmann Equation

$$\boxed{\cancel{\frac{\partial f_{s0}}{\partial t}} + [f_{s0}, H_{s0}] = 0}$$

$$H_0 = \frac{1}{2}\vec{v}^2 + \phi_{s0}(\vec{x})$$

② The Poisson Equation

$$\boxed{\nabla^2 \phi_{s0} = 4\pi G \rho_{s0} = 4\pi G \int d^3v\, f_{s0}(\vec{x},\vec{v},t)}$$

The response of the system to a weak perturbation $\quad - \varepsilon \vec{\nabla} \phi_e$

$$f_{s0}(\vec{x},\vec{v}) \rightarrow f_s(\vec{x},\vec{v},t) = f_{s0}(\vec{x},\vec{v}) + \varepsilon f_{s1}(\vec{x},\vec{v},t)$$

$$\rho_{s0}(\vec{x}) \rightarrow \rho_s(\vec{x},t) = \rho_{s0}(\vec{x}) + \varepsilon \rho_{s1}(\vec{x},t)$$

$$\phi_{s0}(\vec{x}) \rightarrow \phi_s(x,t) = \phi_{s0}(\vec{x}) + \varepsilon \phi_{s1}(\vec{x},t)$$

⚠ system only, without the perturbation

$$\equiv$$

$$\phi_{s0}(\vec{x}) \rightarrow \phi(x,t) = \phi_{s0}(\vec{x}) + \varepsilon \phi_1(\vec{x},t)$$

total pot. include the perturbation

$$= \phi_{s0}(\vec{x}) + \varepsilon \phi_{s1}(\vec{x},t) + \underbrace{\varepsilon \phi_e(\vec{x},t)}_{\text{contrib. of the ext. perturb.}}$$

$$H_0(\vec{x},\vec{v}) \rightarrow H_{tot}(x,t) = \frac{1}{2}\vec{v}^2 + \phi_0(\vec{x}) + \varepsilon \phi_1(\vec{x},t)$$

$$= H_0(\vec{x},\vec{v}) + \varepsilon \phi_1(\vec{x},t)$$

include the perturbation

→ insert those equations into the equations for a self-gravitating stellar system $\quad f_s(\vec{x},\vec{v},t) \quad \rho_s(\vec{x},t) \quad \phi_s(\vec{x},t)$

→ use equations for the unperturbed stellar system

→ keep only first order terms $\quad (\varepsilon^2 \rightarrow 0)$

We get a set of linear differential equations

→ time evolution of the perturbation

① "The collisionless Boltzmann Equation"

f_{s1} ρ_{s1} ϕ_{s1}

3 unknowns

3 equations

$$\frac{\partial f_{s1}}{\partial t} + [\, f_{s1} , H_{s0} \,] = [\, f_{s0} , \phi_1 \,]$$

$$\frac{\partial f_{s1}}{\partial t} + \frac{\partial f_{s1}}{\partial \vec{x}} \vec{v} - \frac{\partial f_{s1}}{\partial \vec{v}} \frac{\partial \phi_{s0}}{\partial \vec{x}} = \frac{\partial f_{s0}}{\partial \vec{v}} \frac{\partial \phi_1}{\partial \vec{x}}$$

② "The Poisson Equation"

$$\nabla^2 \phi_{s1} = 4\pi G \rho_{s1} = 4\pi G \int d^3v \; f_{s1}$$

Interpretation

Reminder Variation of ρ_{s1} along the flow (Lagrangian derivative)

$$\frac{\partial \rho_{s1}}{\partial t} + [\rho_{s1}, H_{s0}] = [\rho_{s0}, \phi_1]$$

$$\underbrace{\frac{\partial \rho_{s1}}{\partial t} + \frac{\partial \rho_{s1}}{\partial \vec{x}}\vec{v} - \frac{\partial \rho_{s1}}{\partial \vec{v}}\frac{\partial \phi_{s0}}{\partial \vec{x}}}_{\frac{d}{dt}\rho_{s1}} = \underbrace{\frac{\partial \rho_{s0}}{\partial \vec{v}}\frac{\partial \phi_1}{\partial \vec{x}}}_{\text{Source}}$$

Source → velocities, gravity

may become big if σ is small

C'est a revoire...

Linear differential equations

$$f(x) \quad : \quad \frac{\partial^n f}{\partial x^n} g_n(x) + \dots + \frac{\partial^2 f}{\partial x^2} g_2(x) + \frac{\partial f}{\partial x} g_1(x) + f(x) g_0(x) = c$$

Ici, q'une var x !!!

$g_i(x)$: a continuous function

if f_1, f_2 are solutions $a f_1 + b f_2$ is a solution

Illustration

- 1-D continuity equation (ρ, v)

Ici, 2 var rho, v !!!

$$\frac{\partial}{\partial t}\rho + \underbrace{\rho \frac{\partial v}{\partial x} + v \frac{\partial \rho}{\partial x}} = 0$$

→ those terms mixes ρ and v

- 1-D linearized continuity equation (ρ_{S1}, v_1)

$$\frac{\partial}{\partial t}\rho_{S1} + \rho_0 \frac{\partial v_1}{\partial x} + v_0 \frac{\partial \rho_{S1}}{\partial x} = 0$$

→ no mixing

Stability of collisionless systems

The Jeans instability

(1902)

The Jeans instability

$$\boxed{\text{Homogeneous medium subject to a perturbation}}$$

- consider an infinite static homogeneous system at equilibrium
  - ⚠ no polytropic homogeneous system exists

$$\boxed{\vec{V}_0 = 0}$$

Euler Equation

$$\cancel{\frac{\partial \vec{v}_0}{\partial t}} + \cancel{(\vec{v} \cdot \vec{\nabla})\vec{v}_0} = - \cancel{\frac{\vec{\nabla} P_0}{\rho_0}} - \vec{\nabla}\phi_0$$

$$\Rightarrow \quad \vec{\nabla}\phi_0 = 0$$

Poisson Equation

$$\nabla^2 \phi_0 = 4\pi G \rho_0 \quad \Rightarrow \quad \underline{\rho_0 = 0}$$

Linearized Poisson

$$\nabla^2 \phi_s = 4\pi G \rho_s = \nabla^2(\phi_0 + \varepsilon \phi_{s1}) = 4\pi G (\rho_0 + \varepsilon \rho_{s1})$$

$$\boxed{\nabla^2 \phi_{s1} = 4\pi G \rho_{s1}} \qquad \text{Jeans swindle}$$

The response of an homogeneous fluid

Linearized fluid equations

$\rho_{s1}(\vec{x},t)$, $\phi_{s1}(\vec{x},t)$, $\vec{V}_{s1}(x,t)$, $h_{s1}(\vec{x},t)$

$$\boxed{\vec{V}_0 = 0}$$

$$\begin{cases}
① \quad \dfrac{\partial}{\partial t}\rho_{s1} + \vec{\nabla}(\rho_0 \vec{V}_{s1}) + \cancel{\vec{\nabla}(\rho_{s1}\vec{V}_0)} = 0 \\
② \quad \dfrac{\partial}{\partial t}\vec{V}_{s1} + \cancel{\vec{V}_0\cdot(\vec{\nabla}\cdot\vec{V}_{s1})} + \cancel{\vec{V}_{s1}(\vec{\nabla}\cdot\vec{V}_0)} = -\vec{\nabla}(h_{s1} + \phi_{s1} + \phi_e) \\
③ \quad \vec{\nabla}\phi_{s1} = 4\pi G\rho_{s1} \\
④ \quad h_{s1} = V_s^2 \dfrac{\rho_{s1}}{\rho_0}
\end{cases}$$

Solution

$$\frac{\partial}{\partial t}① : \quad \frac{\partial^2}{\partial t^2}\rho_{s1} + \rho_0 \vec{\nabla}\cdot\left(\frac{\partial}{\partial t}\vec{V}_{s1}\right) = 0$$

$$\vec{\nabla}\cdot② : \quad \vec{\nabla}\cdot\frac{\partial \vec{V}_{s1}}{\partial t} = -\nabla^2(h_{s1} + \phi_{s1} + \phi_e)$$

Thus, we have $\quad$ ($\vec{\nabla}\cdot②$ into $\frac{\partial}{\partial t}①$)

$$\frac{\partial^2}{\partial t^2}\rho_{s1} - \rho_0 \nabla^2(h_{s1} + \phi_{s1} + \phi_e) = 0$$

$$\frac{\partial^2}{\partial t^2}\rho_{s1} - \rho_0 \nabla^2 \left(h_{s1} + \phi_{s1} + \phi_e \right) = 0$$

(*) $\vec{\nabla} h_{s1} = \frac{1}{\rho_s} \vec{\nabla} \rho_{s1}$

$$\frac{\partial^2}{\partial t^2}\rho_{s1} - \rho_0 \nabla^2 h_{s1} - \rho_0 \nabla^2 \phi_{s1} = \rho_0 \nabla^2 \phi_e$$

$$\boxed{\underbrace{\frac{1}{\rho_0}\frac{\partial^2}{\partial t^2}\rho_{s1}}_{\text{growth accel. of } \rho_{s1}} - \underbrace{\vec{\nabla}\cdot\left(\frac{\vec{\nabla}\rho_{s1}}{\rho_s}\right)}_{F_{\text{pressure}}} - \underbrace{\vec{\nabla}\cdot\left(\vec{\nabla}\phi_{s1}\right)}_{F_{\text{grav, self}}} = \underbrace{\vec{\nabla}\cdot\left(\vec{\nabla}\phi_e\right)}_{F_{\text{grav, ext}}}}$$

(spec. forces)

$$\frac{\partial^2}{\partial t^2}\rho_{s1} \underbrace{- \rho_0 \nabla^2 h_{s1}}_{(4)\; - v_s^2 \nabla^2 \rho_{s1}} \underbrace{- \rho_0 \nabla^2 \phi_{s1}}_{(3)\; - 4\pi G \rho_0 \rho_{s1}} = \underbrace{\rho_0 \nabla^2 \phi_e}_{4\pi G \rho_0 \rho_e}$$

$\nabla^2 \phi_e = 4\pi G \rho_e$

$$\boxed{\frac{\partial^2}{\partial t^2}\rho_{s1} \underbrace{- v_s^2 \nabla^2 \rho_{s1}}_{\text{pressure}} \underbrace{- 4\pi G \rho_0 \rho_{s1}}_{\text{self grav.}} = \underbrace{4\pi G \rho_0 \rho_e}_{\text{ext. grav.}}}$$

Differential equation for ρ_{s1} only

Evolution of a perturbation in an homogeneous fluid

Some simple cases

$$\boxed{\frac{\partial^2}{\partial t^2}\rho_{s1} - v_s^2 \nabla^2 \rho_{s1} - 4\pi G \rho_0 \rho_{s1} = 4\pi G \rho_0 \rho_e}$$

1) no gravity, no ext forces

$$\boxed{\frac{\partial^2}{\partial t^2}\rho_{s1} - v_s^2 \nabla^2 \rho_{s1} = 0}$$

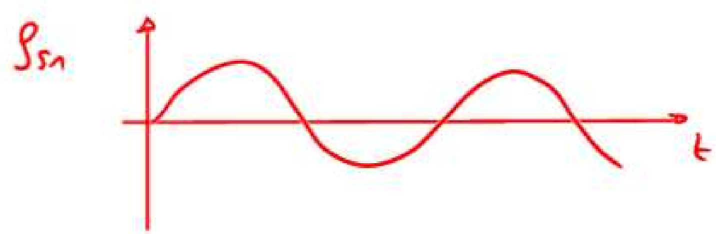
In 1-D

$$\frac{\partial^2}{\partial t^2}\rho_{s1}(x,t) - v_s^2 \frac{\partial^2}{\partial x^2}\rho_{s1}(x,t) = 0 \quad \text{(wave equation)}$$

general solution : f : an arbitrary function

$$\rho_{s1}(x,t) = \underbrace{f(x - v_s t)}_{\text{more at speed } v_s} + \underbrace{f(x + v_s t)}_{\text{more at speed } -v_s}$$

sound wave traveling at sound speed v_s



oscillatory solution

STABLE

$$\boxed{\frac{\partial^2}{\partial t^2}\rho_{s1} - v_s^2\nabla^2\rho_{s1} - 4\pi G\rho_0\rho_{s1} = 4\pi G\rho_0\rho_e}$$

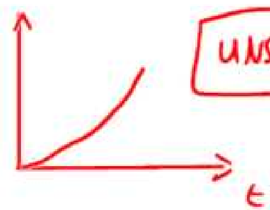
2) <u>no pressure, no external forces</u>

$$\boxed{\frac{\partial^2}{\partial t^2}\rho_{s1} = 4\pi G\rho_0\,\rho_{s1}}$$

<u>Solutions</u> : any linear combinaison of

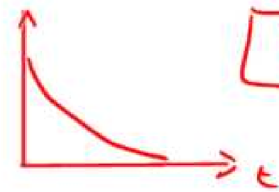
$$\rho_{s1}^{(1)}(x,t) = \tilde{\rho}(\vec{x})\, e^{\sqrt{4\pi G\rho_0}\,t} \quad \text{and} \quad \rho_{s1}^{(2)}(x,t) = \tilde{\rho}(\vec{x})\, e^{-\sqrt{4\pi G\rho_0}\,t}$$

$\rho_{s1}^{(1)}$



UNSTABLE

$\rho_{s1}^{(2)}$



DAMPED

3) <u>no external forces</u>

$$\boxed{\frac{\partial^2}{\partial t^2}\rho_{s1} - v_s^2\nabla^2\rho_{s1} - 4\pi G\rho_0\rho_{s1} = 0}$$

pressure: <u>stabilize</u>

self-gravity: <u>destabilize</u>

General solutions

$$\boxed{e^{-i(\vec{k}\vec{x}-\omega t)}}$$

- $\times\ e^{-i\vec{k}\vec{x}} \;+\; \int d^3\vec{x} \;\Rightarrow\;$ spatial Fourier transform

$$\bar{\rho}_{sn}(\vec{k},t) = \int d^3x\, e^{-i\vec{k}\vec{x}}\, \rho_{sn}(\vec{x},t)$$

$$\boxed{\frac{\partial^2}{\partial t^2}\bar{\rho}_{sn}(\vec{k},t) + v_s^2 k^2\, \bar{\rho}_{sn}(\vec{k},t) - 4\pi G\rho_0\, \bar{\rho}_{sn}(\vec{k},t) = 4\pi G\rho_e\, \bar{\rho}_e(\vec{k},t)}$$

Evolution in absense of perturbation (after the) $\quad \rho_e(x,t)=0 \quad \bar{\rho}_e = 0$

Solutions are in the form $\quad \bar{\rho}_{sn}(\vec{k},t) \sim e^{i\omega t} \quad \omega = \omega(|\vec{k}|) \in \mathbb{C}$, $\quad k = |\vec{k}|$

Introducing in the previous equation, we get the dispersion relation

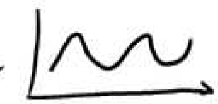
$$\boxed{\omega^2(k) = v_s^2 k^2 - 4\pi G\rho_0}$$

<u>Definition</u> : Jeans wave number $\boxed{k_J^2 = \dfrac{4\pi G \rho_0}{v_s^2}}$

$$\boxed{\omega^2(k) = v_s^2 k^2 - 4\pi G \rho_0} \rightarrow \boxed{\omega^2(k) = v_s^2 (k^2 - k_J^2)}$$

$\rho_{s1} \sim e^{ikx} e^{-i\omega t} \quad (k \in \mathbb{R}e)$

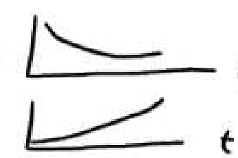
- <u>If $k > k_J$</u> $\rightarrow \omega^2 > 0 \rightarrow \omega \in \mathbb{R}e \qquad \bar{\rho}_{s1} \sim e^{-i\omega t} \qquad \bar{\rho}_{s1} \sim$



A perturbation with a short wavelenght ($k > k_J$) will see its amplitude oscillate

$\omega' = i\omega \in \mathbb{R}e$

- <u>If $k < k_J$</u> $\rightarrow \omega^2 < 0 \rightarrow \omega \in \mathbb{I}m \qquad \bar{\rho}_{s1} \sim e^{-\omega' t} \qquad \bar{\rho}_{s1} \sim$



A perturbation with a long wavelenght ($k < k_J$) will see its amplitude decay or growth exponentially

More discussion

$$k_J^2 = \frac{4\pi G \rho_0}{v_s^2}$$

$$\omega^2(k) = v_s^2 (k^2 - k_J^2)$$

Assume a perturbation with a fixed wave number k.

① $G = 0$ (no gravity) $\Rightarrow$ $k_J = 0$ — $k > k_J$

$$\omega^2(k) = v_s^2 k^2 \in \mathbb{R}_e$$

dispersion relation for a sound wave

$$\bar{\rho}_{s_1} \sim e^{-i\omega t}$$

oscillation at frequ. ω

$\boxed{\text{STABLE}}$

② $G \neq 0$

$k > k_J$ — ω decreases with k increases — $\boxed{\text{STABLE}}$

$k = k_J$ — $\omega = 0$

$k < k_J$ — $\omega \in I_m$ — $\boxed{\text{UNSTABLE}}$

③ $\rho \gg v_s^2 / 4\pi G$ (no pressure) $k_J \to \infty$ — $\boxed{\text{UNSTABLE}}$

Definitions

Jeans length

$$\boxed{\lambda_J = \frac{2\pi}{k_J} = \sqrt{\frac{\pi v_s^2}{G\rho_0}}}$$

$$\lambda > \lambda_J \Rightarrow \boxed{\text{UNSTABLE}}$$

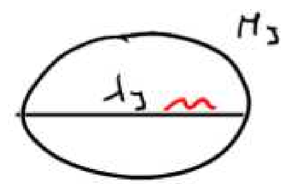
Jeans mass

Mass inside a sphere of radius $\frac{1}{2}\lambda_J$

$$M_J = \frac{4}{3}\pi\left(\frac{1}{2}\lambda_J\right)^3\rho_0$$

$$\boxed{M_J = \frac{\pi^{5/2}}{6}\,\frac{v_s^3}{G^{3/2}\rho_0^{1/2}}}$$

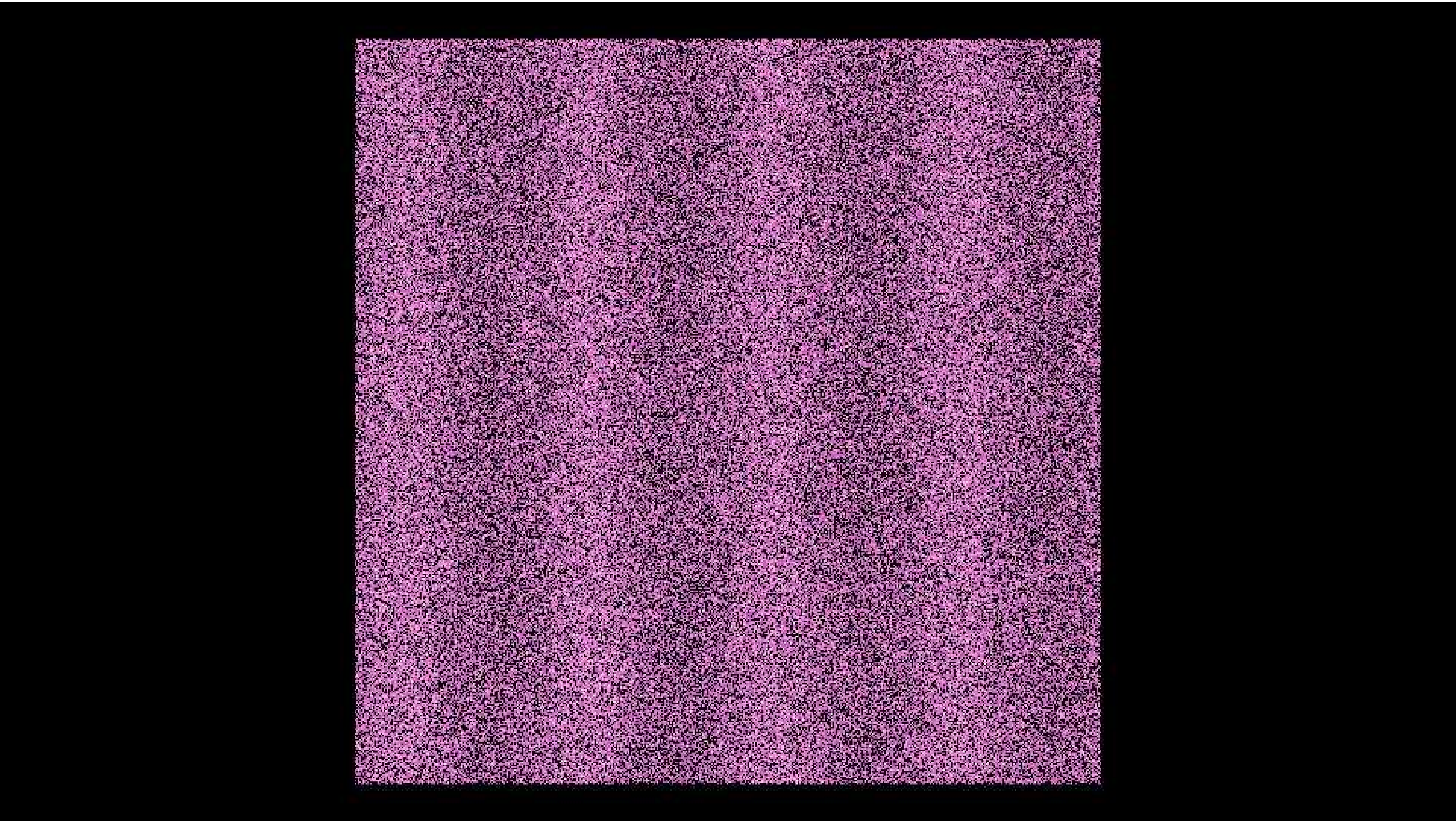
"Minimum mass that can collapse"



any perturbation with $\lambda < \lambda_J$ i.e. involving a mass $< M_J$ will be STABLE

The Jeans instability in fluid systems

$\lambda_{\mathrm{J}} = 1.50 \ \lambda = 0.25$



The Jeans instability in fluid systems

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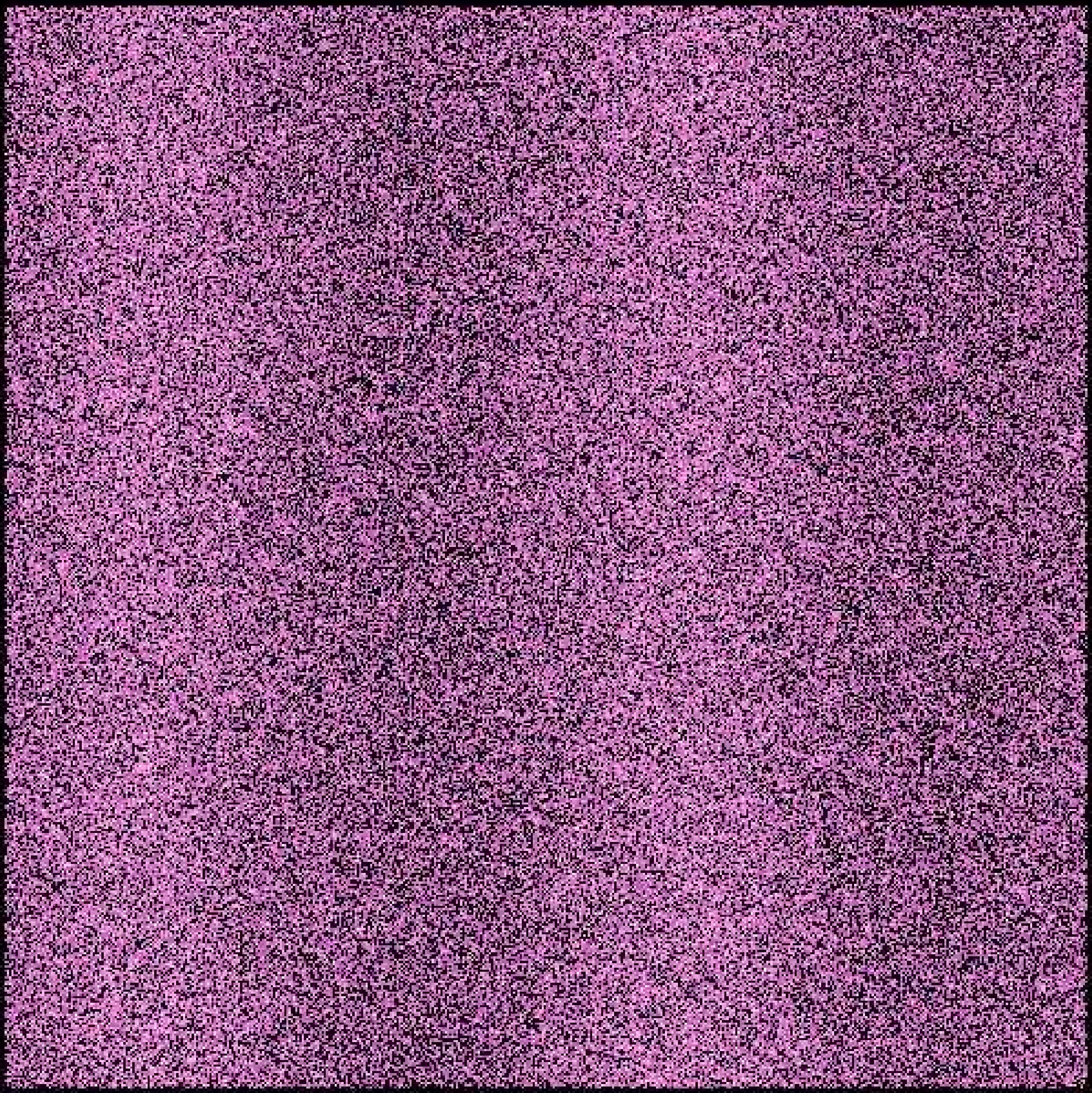

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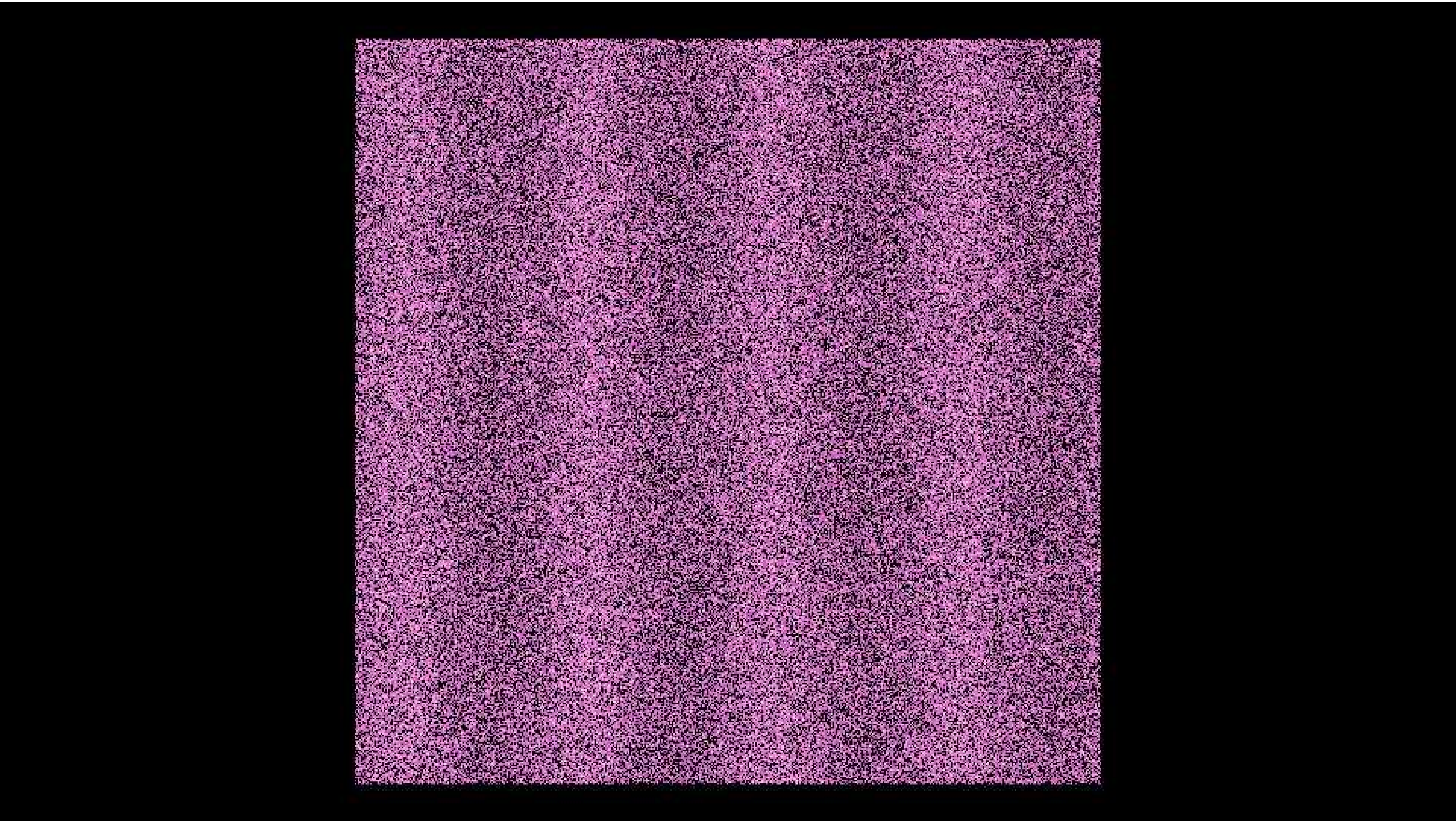

The Jeans instability in fluid systems

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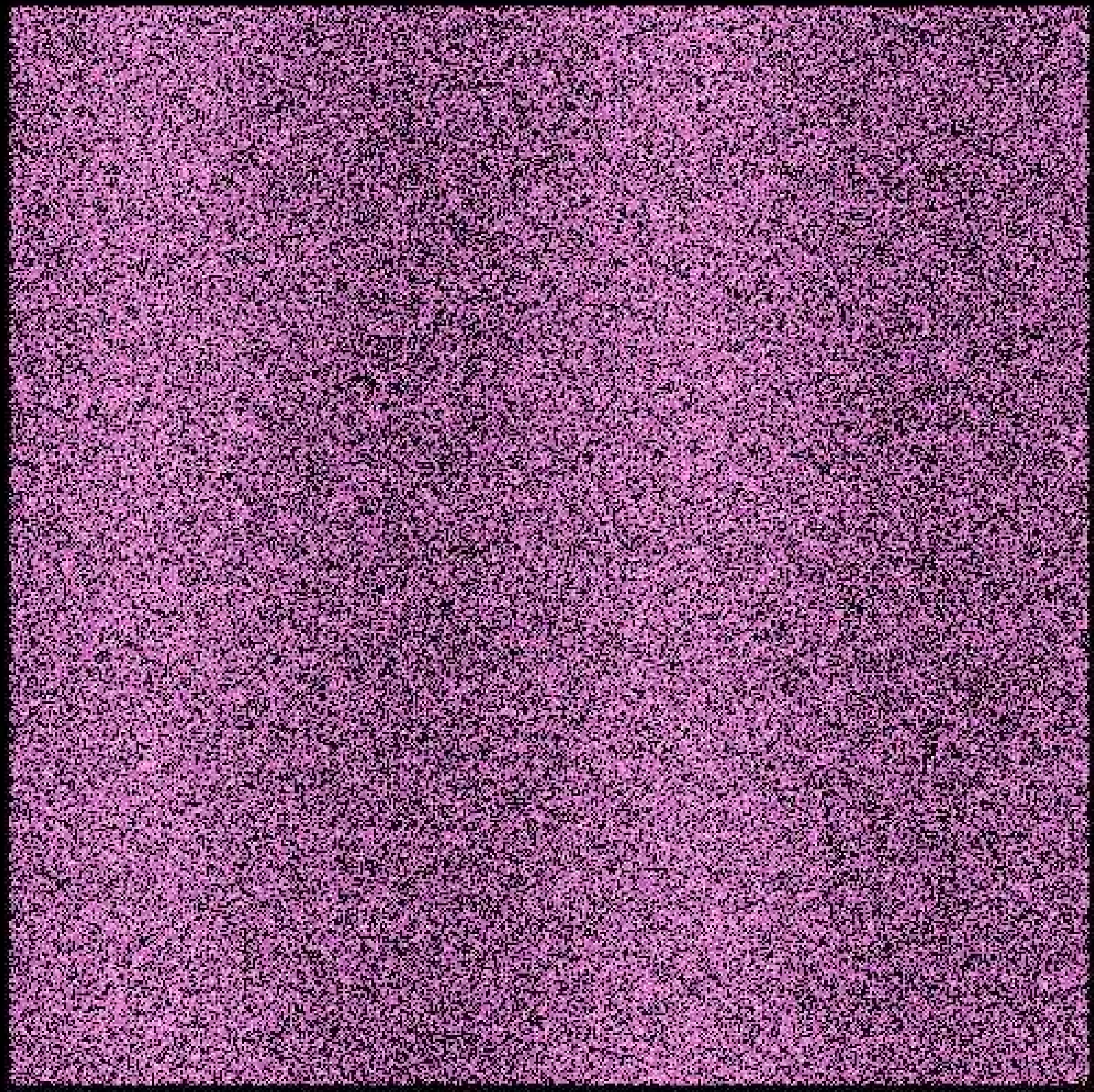


The response of an homogeneous stellar system

Linearized equations

$$
\begin{cases}
① \quad \dfrac{\partial f_{s1}}{\partial t} + [f_{s1}, H_0] = [f_0, \phi_1] \\[2ex]
\equiv \dfrac{\partial f_{s1}}{\partial t} + \dfrac{\partial f_{s1}}{\partial \vec{x}}\vec{v} - \cancel{\dfrac{\partial f_{s1}}{\partial \vec{v}}\dfrac{\partial \phi_0}{\partial \vec{x}}}_{\phi_0 = 0} = \dfrac{\partial f_0}{\partial \vec{v}}\dfrac{\partial \phi_1}{\partial \vec{x}} \equiv \dfrac{\partial f_0}{\partial \vec{v}}\dfrac{\partial}{\partial \vec{x}}\left(\phi_{s1} + \phi_e\right) \\[2ex]
② \quad \nabla^2 \phi_{s1} = 4\pi G \rho_{s1} = 4\pi G \int d^3\vec{v}\, f_{s1}
\end{cases}
$$

Manipulation + spacial Fourier space

$$
\bar{f}_{s1}(\vec{k},\vec{v},t) = i\vec{k}\cdot\frac{\partial f_0}{\partial \vec{v}} \int_{-\infty}^{t} dt'\, e^{i\vec{k}\vec{v}(t'-t)} \left[\bar{\phi}_{s1}(\vec{k},t') + \bar{\phi}_e(\vec{k},t')\right]
$$

⚠ The DF depends on the past history $\left(\int_{-\infty}^{t} \phi_{s1} + \phi_e \right)$

Integration over $d^3\vec{v}$ + Poisson

$$\bar{\rho}_{s1}(\bar{k},t) = -\frac{4\pi G}{k^2} i \int d^3\bar{v}\, \bar{k}\cdot\frac{\partial f_0}{\partial \bar{v}} \int_{-\infty}^{t} dt'\, e^{i\vec{k}\cdot\bar{v}(t'-t)} \left[\bar{\rho}_{s1}(k,t') + \bar{\rho}_e(k,t') \right]$$

In temporal Fourier space

$$\tilde{\bar{\rho}}_{s1}(\bar{k},\omega) = \int dt\, \bar{\rho}_{s1}(\bar{k},t)\, e^{-i\omega t}$$

this term may diverge if $\bar{k}\cdot\bar{v} = \omega$

$$\tilde{\bar{\rho}}_{s1}(\bar{k},\omega) = \left(-\frac{4\pi G}{k^2} \int \frac{d^3\bar{v}}{\bar{k}\cdot\bar{v} - \omega}\, \bar{k}\cdot\frac{\partial f_0}{\partial \bar{v}} \right) \left(\tilde{\bar{\rho}}_{s1}(\bar{k},\omega) + \hat{\bar{\rho}}_e(\vec{k},\omega) \right)$$

In absense of perturbation

$\rho_e = 0$

we must have:

$$\boxed{-\frac{4\pi G}{k^2} \int \frac{d^3\bar{v}}{\bar{k}\cdot\bar{v} - \omega}\, \bar{k}\cdot\frac{\partial f_0}{\partial \bar{v}} = 1}$$

$\mathrm{Im}(\omega) > 0$

(instead, we may have a divergence)

This is our dispertion relation

$$\omega = \omega(\bar{k}, f_0)$$

Assuming a Maxwellian for the unperturbed DF f_0

$$\boxed{f_0(\bar{v}) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}} e^{-\frac{v^2}{2\sigma^2}}}$$

The dispersion relation becomes

$$\boxed{-\frac{4\pi G}{k^2}\int \frac{d^3\bar{v}}{\bar{k}\cdot\bar{v}-\omega}\,\bar{k}\cdot\frac{\partial f_0}{\partial\bar{v}} = 1} \Rightarrow \boxed{\frac{4\pi G\rho_0}{k^2\sigma^2}\left(1 + w' Z(w')\right) = 1}$$

with

$$w' = \sqrt{2}\,k\sigma w \qquad Z(w') = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} ds\,\frac{e^{-s^2}}{s-w'} = i\sqrt{\pi}\,e^{-w'^2}\left[1 + \mathrm{erf}(iw')\right]$$

$$k = |k|$$

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z dt\, e^{-t^2}$$

The dispersion relation is $\quad Im(w) > 0$

$$\boxed{\frac{4\pi G \rho_0}{k^2 \sigma^2}\left(1 + w' Z(w')\right) = 1}$$

$$w' = \sqrt{2}\, k \sigma w$$

$$Z(w') = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} ds \, \frac{e^{-s^2}}{s - w'}$$

Defining $\quad k_J^2 = \frac{4\pi G \rho_0}{\sigma^2}$

$$\boxed{\frac{k^2}{k_J^2} = 1 + w' \cdot Z(w')}$$

$$\boxed{\frac{w^2}{\xi(w')} = \sigma^2 (k^2 - k_J^2)}$$

one k is hidden here

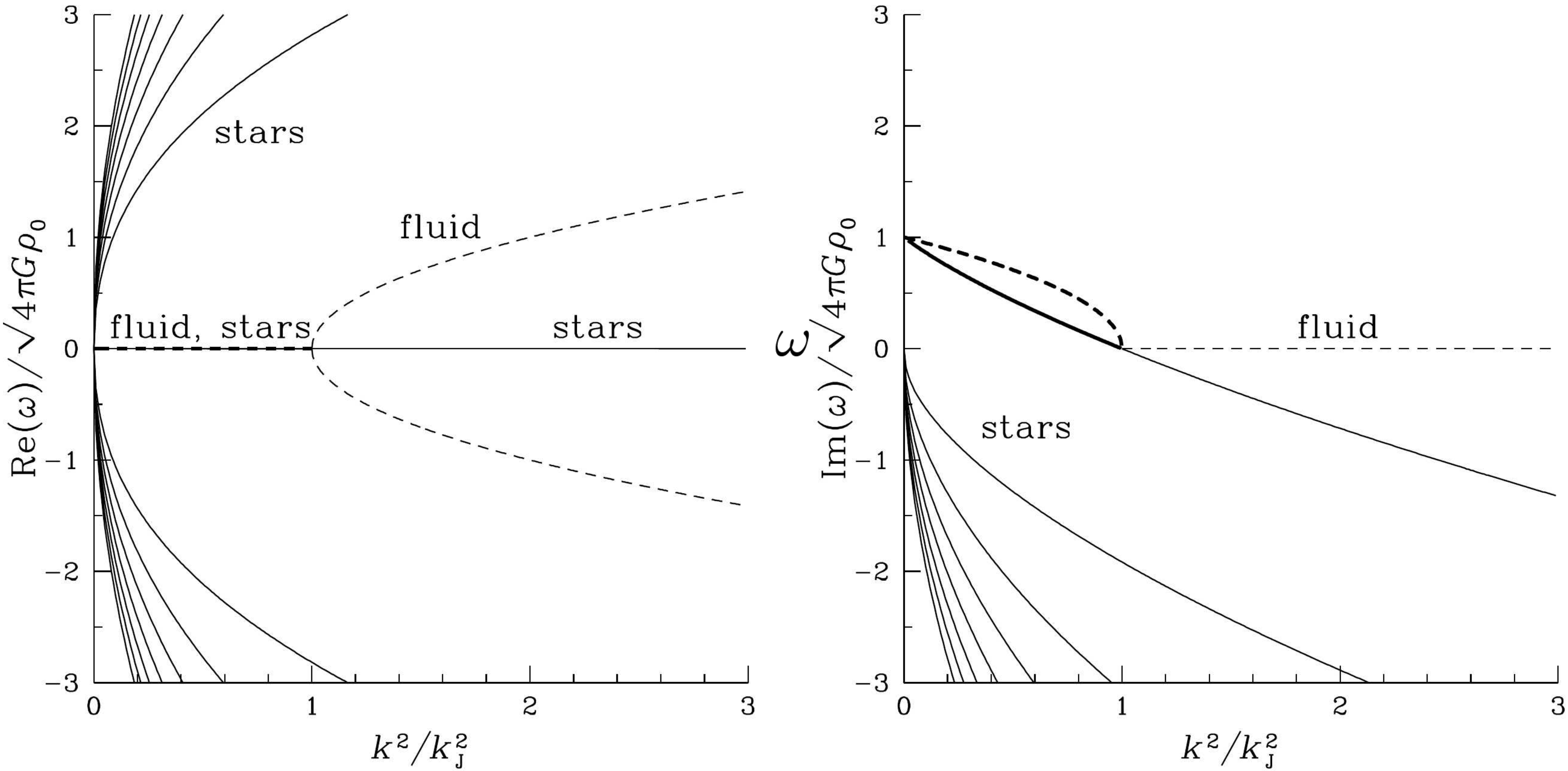
fluid $\quad k_J^2 = \frac{4\pi G \rho_0}{c_s^2}$

$$\boxed{\frac{k^2}{k_J^2} = 1 + \frac{w^2}{k_J^2 c_s^2}}$$

$$\boxed{w^2 = c_s^2 (k^2 - k_J^2)}$$

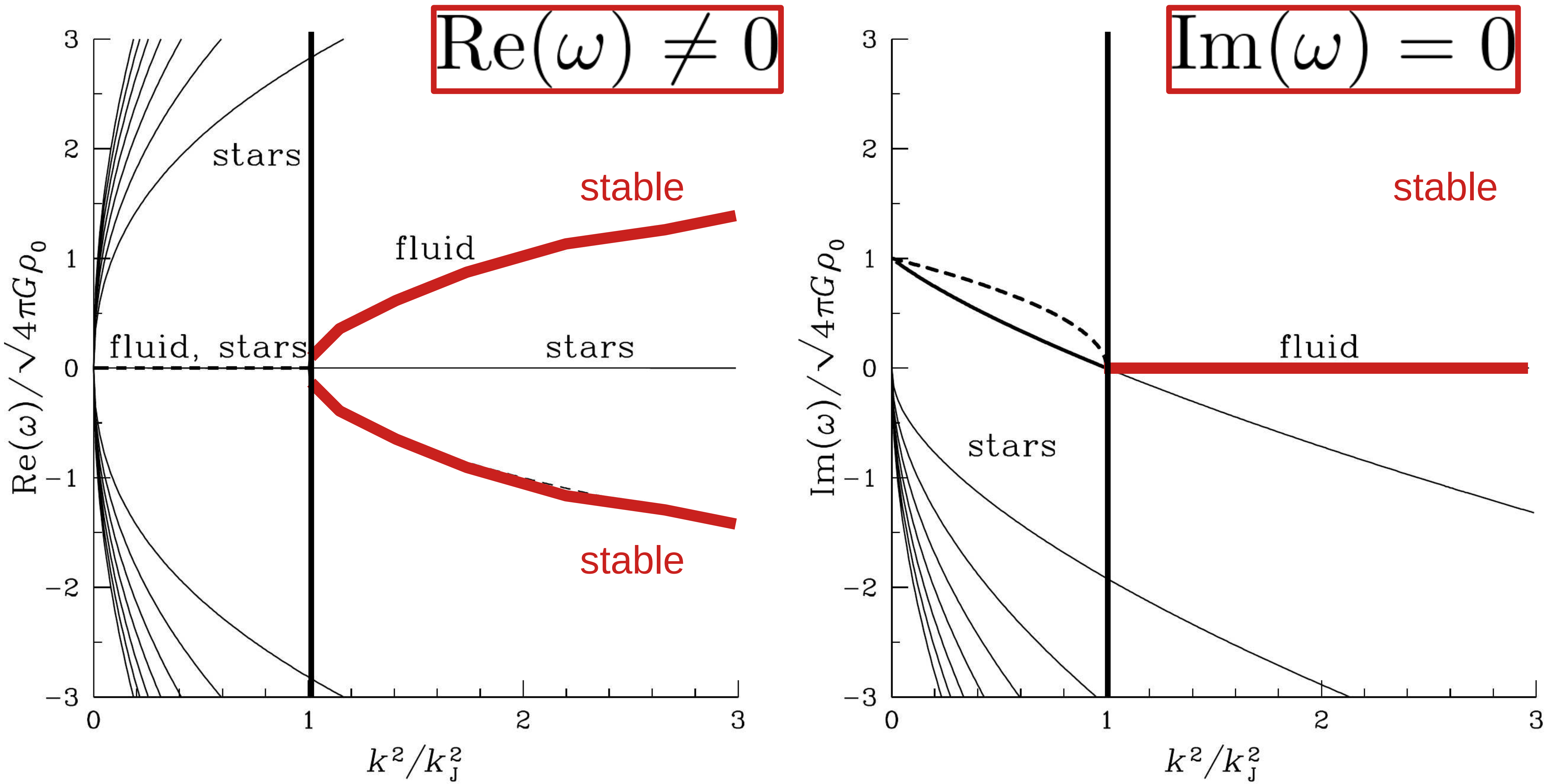
! for $\quad Im(w) = 0 \quad$ or $\quad Im(w) < 0 \quad$ (a bit more tricky)

The dispersion relation for fluids and stellar systems



The dispersion relation for fluids

$$\frac{k^2}{k_J^2} = 1 + \frac{\omega^2}{k_J^2 v_s^2}$$



The dispersion relation for fluids

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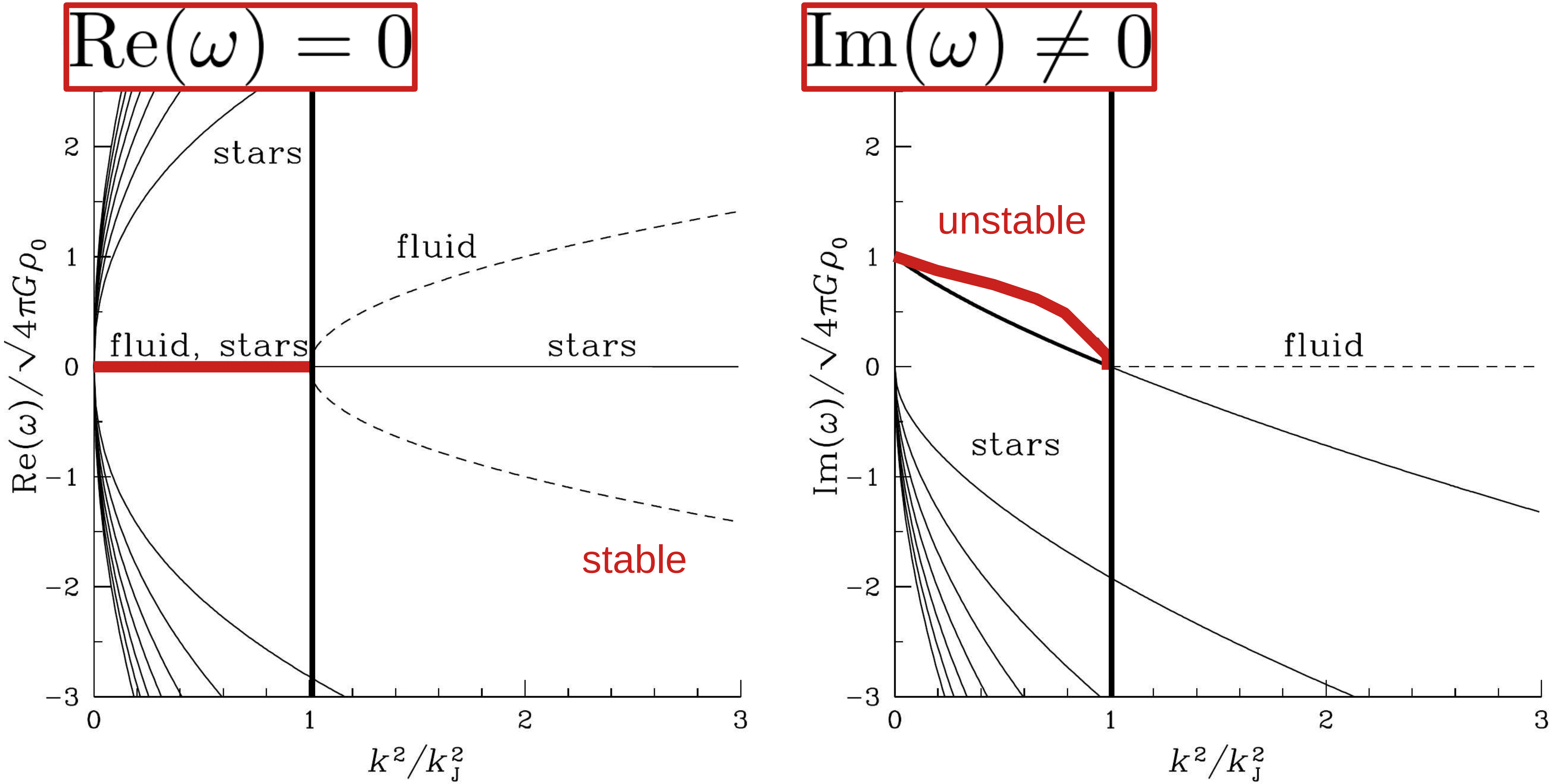
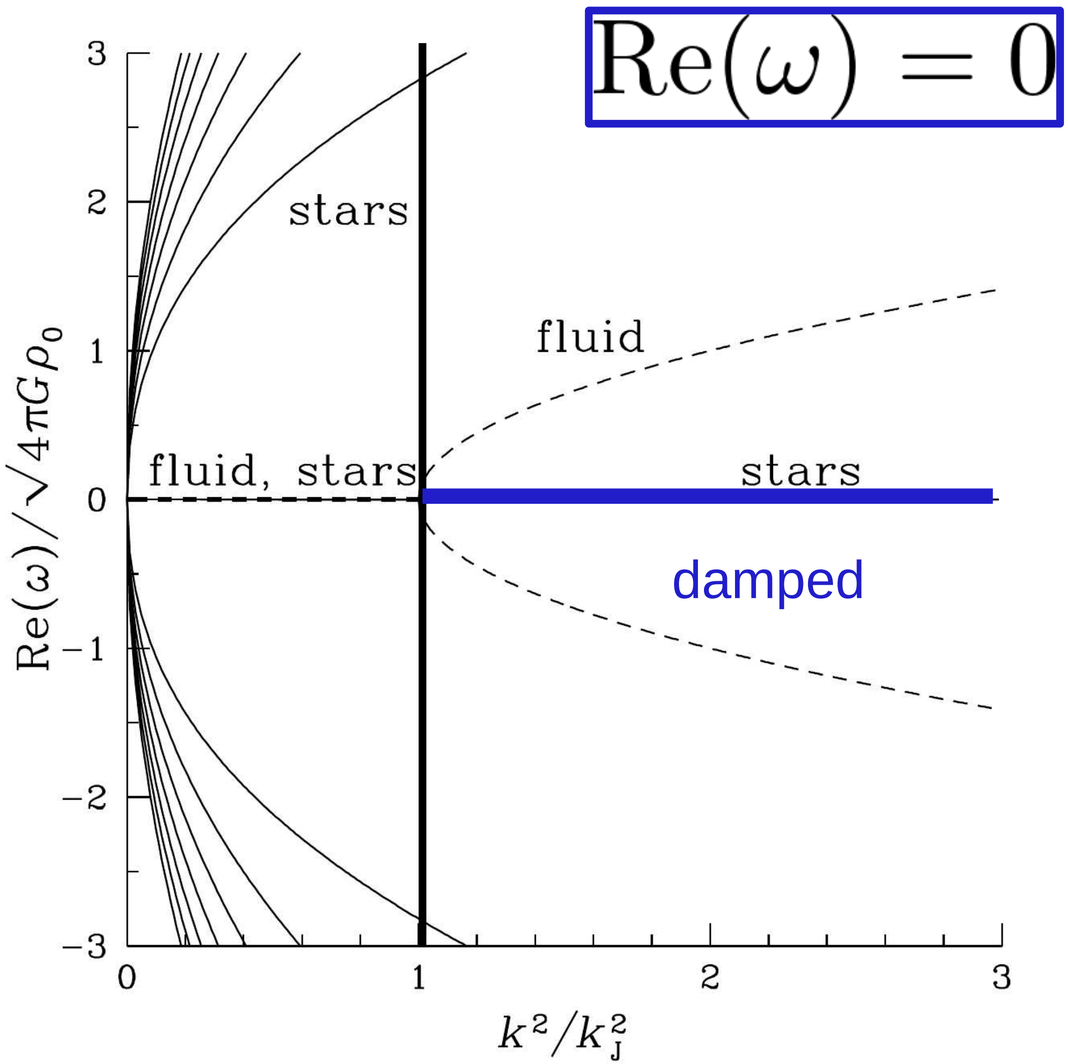
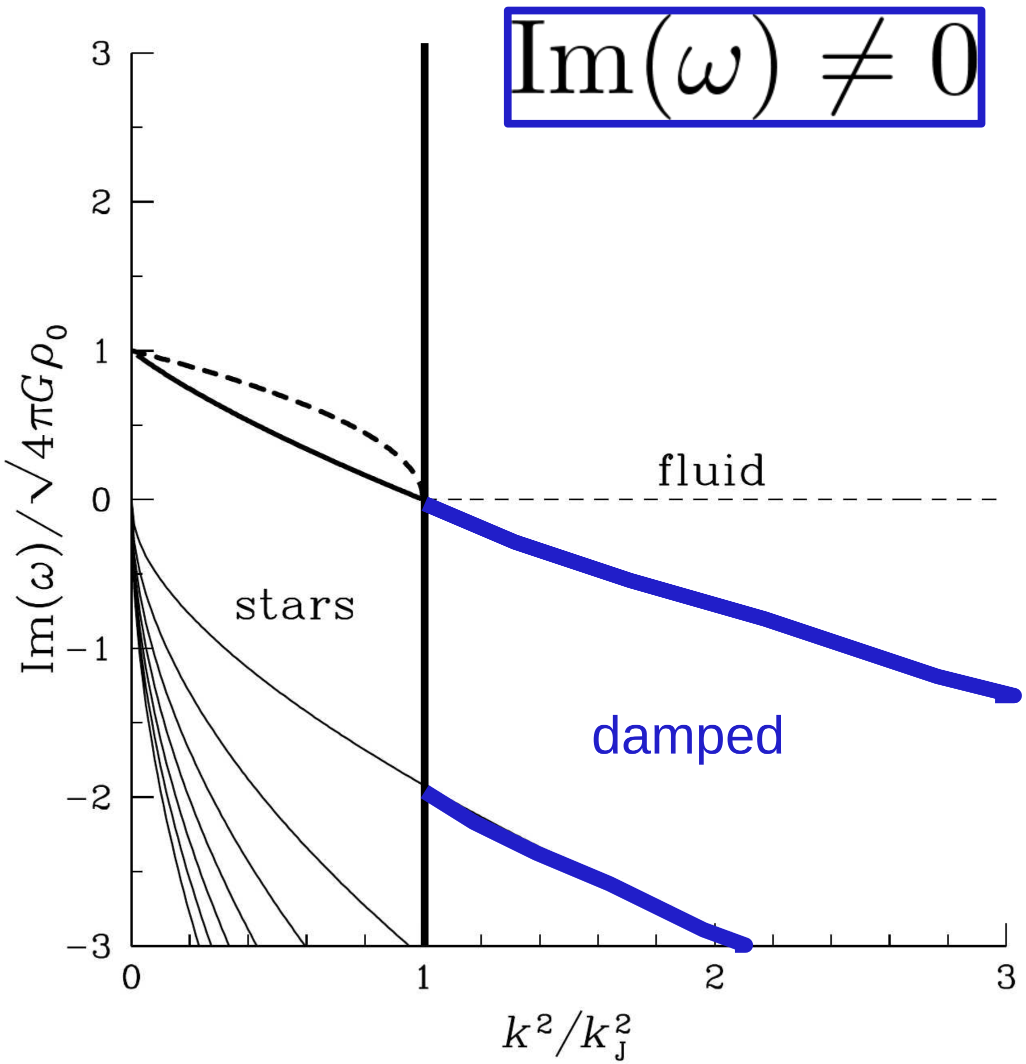


The dispersion relation for stellar systems

$$\frac{k^2}{k_J^2} = 1 + w' \cdot Z(w')$$





The dispersion relation for stellar systems

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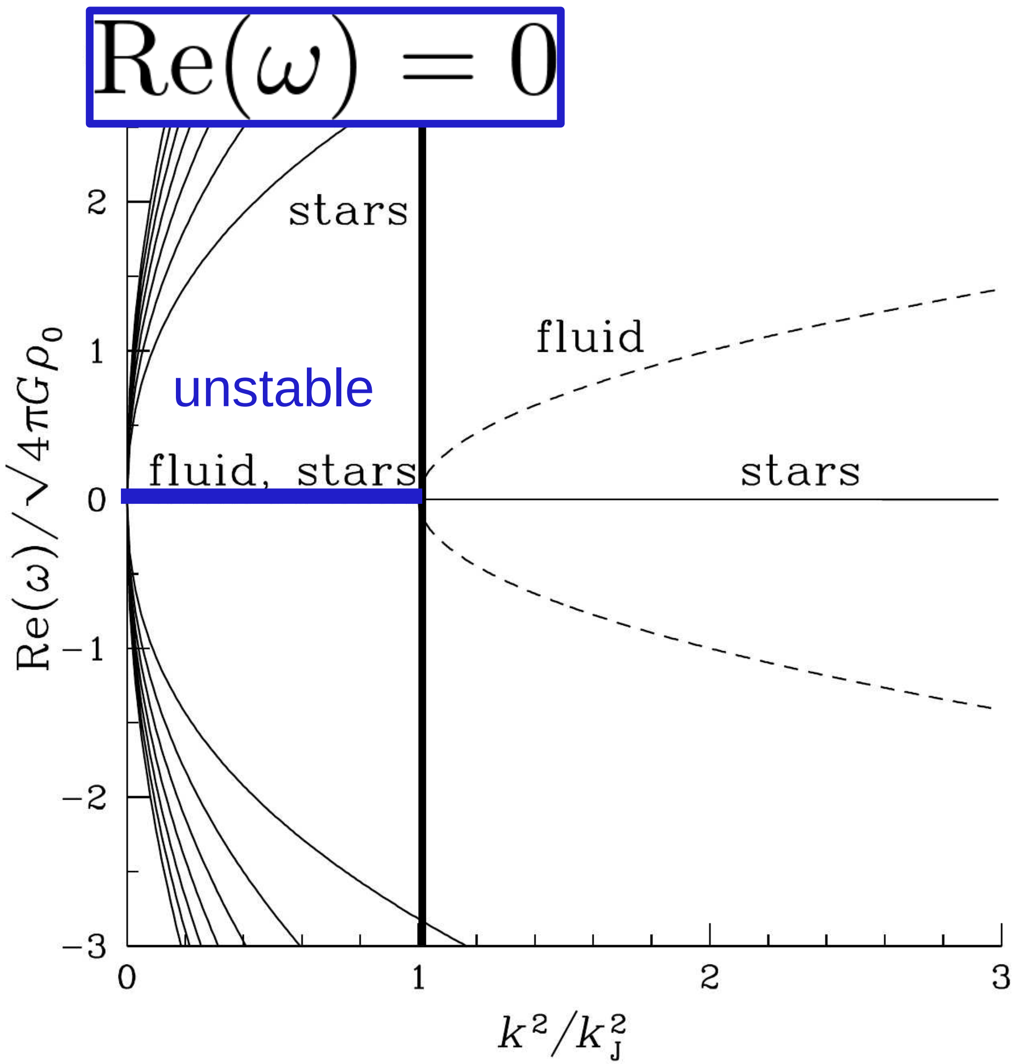
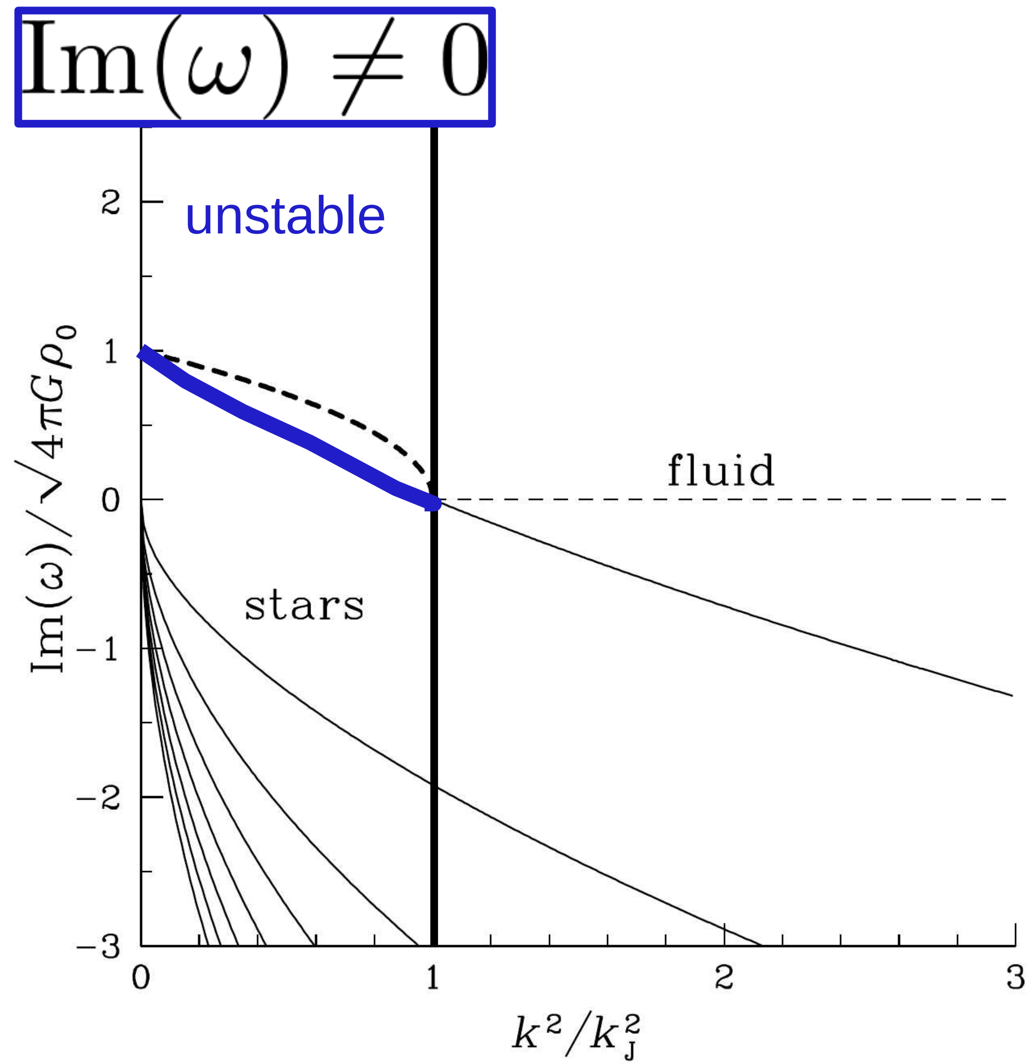
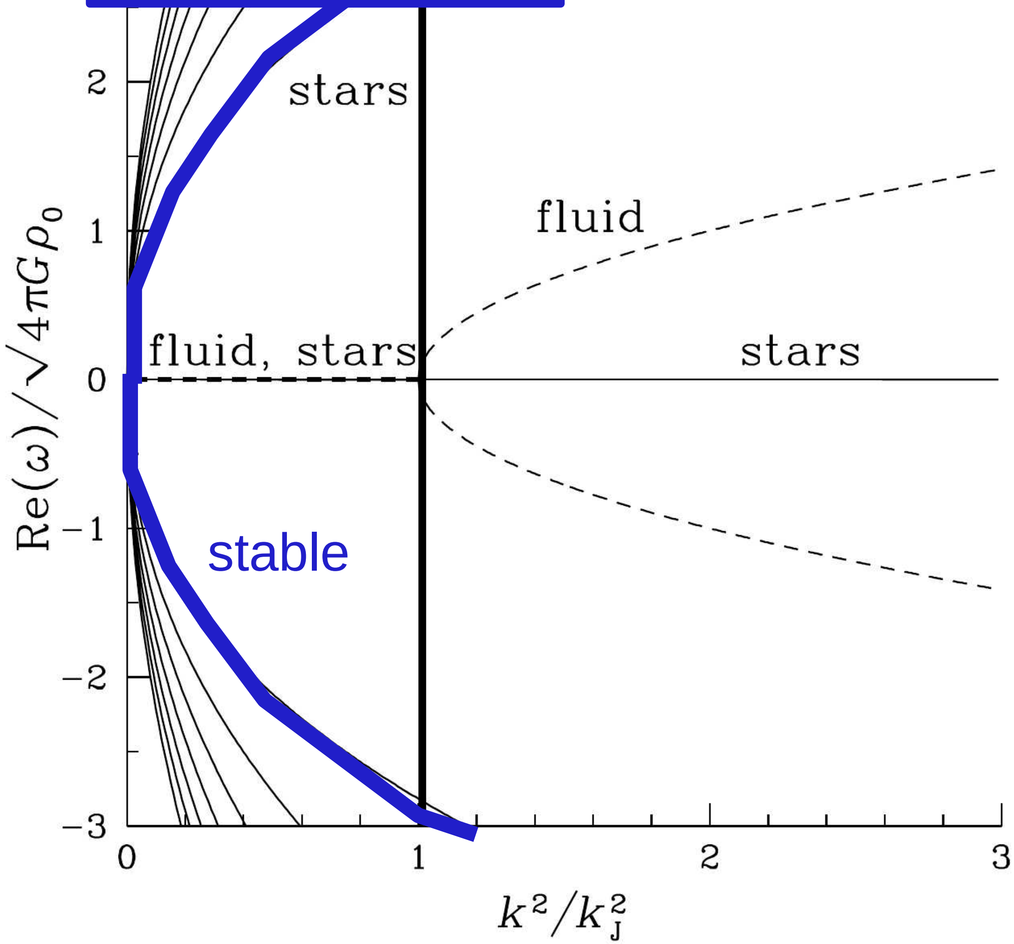




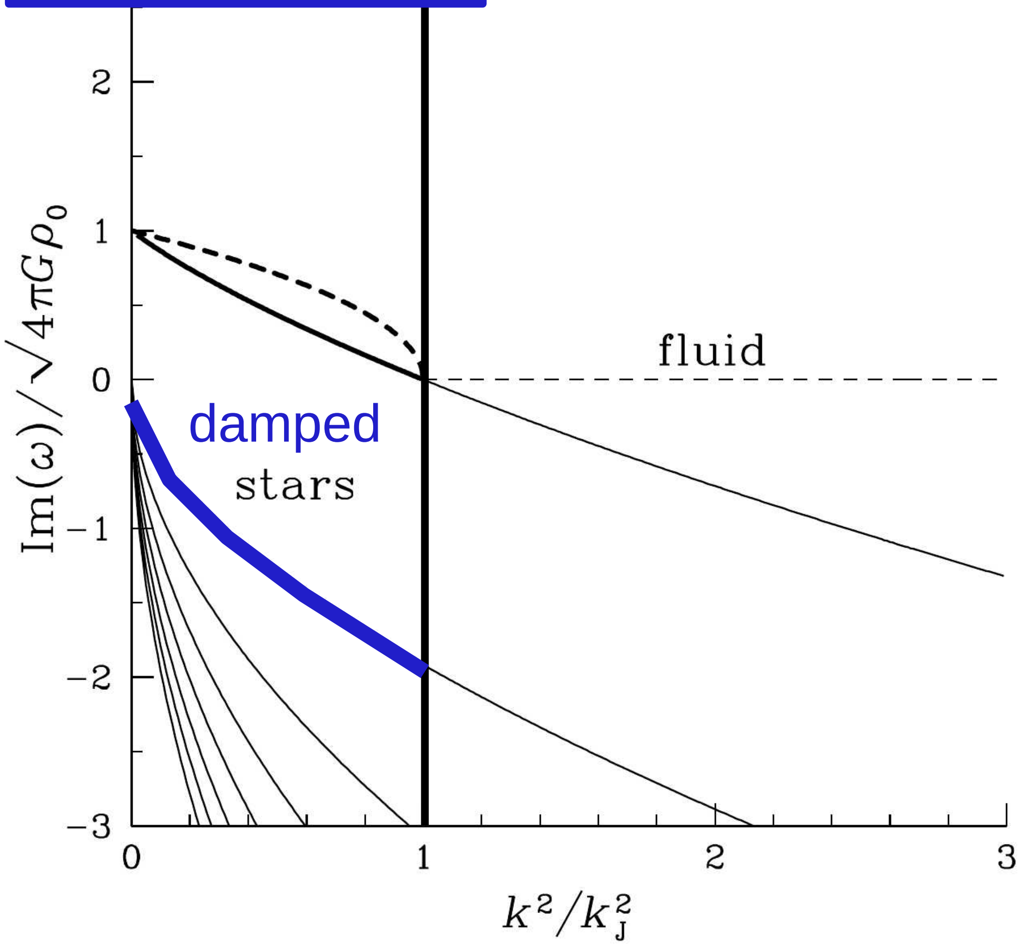
The dispersion relation for stellar systems

$$\frac{k^2}{k_J^2} = 1 + w' \cdot Z(w')$$

$\mathrm{Re}(\omega) \neq 0$



$\mathrm{Im}(\omega) \neq 0$



Stability

$$\boxed{\tilde{\rho} \sim e^{-i\omega t}}$$

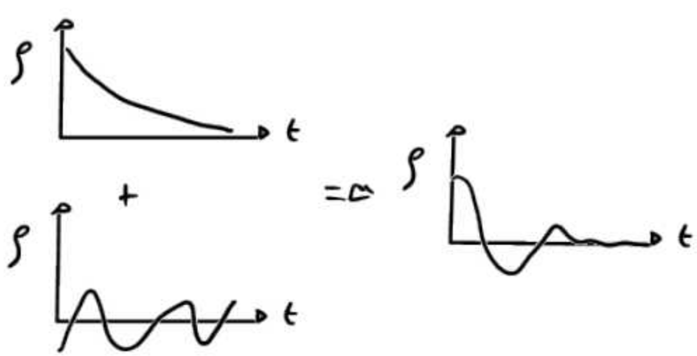
Ⓐ **Unstable** $\quad \mathrm{Im}(\omega) > 0, \; \mathrm{Re}(\omega) = 0$

$\underline{k^2 < k_J^2}$ (same than hydro)

Ⓑ **Stable** $\quad \mathrm{Im}(\omega) = 0, \; \mathrm{Re}(\omega) \neq 0$

$\underline{\omega = 0 \; , \; k = k_J}$ $\Rightarrow$ **no stable solution**

Ⓒ **Damped** $\quad \mathrm{Im}(\omega) < 0$



→ plenty of damped solutions

(with or without oscillation)

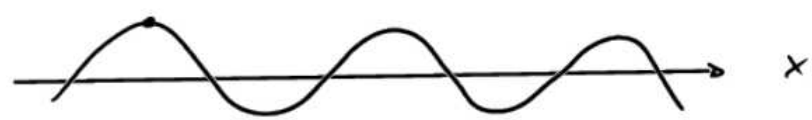
Landau damping

The existance of solutions with $Im(\omega) < 0$ is due to

$$\tilde{\tilde{\rho}}_{sn}(\vec{k},\omega) = \left(-\frac{4\pi G}{k^2}\int \frac{d^3\vec{v}}{\boxed{\vec{k}\cdot\vec{v}-\omega}}\,\vec{k}\cdot\frac{\partial \rho_0}{\partial \vec{v}}\right)\left(\tilde{\tilde{\rho}}_{sn}(\vec{k},\omega)+\hat{\hat{\rho}}_e(\vec{k},\omega)\right)$$

Wave in 1-D

$$\rho \sim \rho(kx-\omega t)$$



position / velocity of a static point with respect to the wave

$$x = \frac{\omega t}{k} + ct \qquad v = \frac{\omega}{k} \qquad \Rightarrow \quad \boxed{kv - \omega = 0}$$

⇒ the integral diverges for particles moving at the same velocity than the wave

Interpretation: Density wave (perturbation) with a speed $v_w = \frac{\omega}{k}$

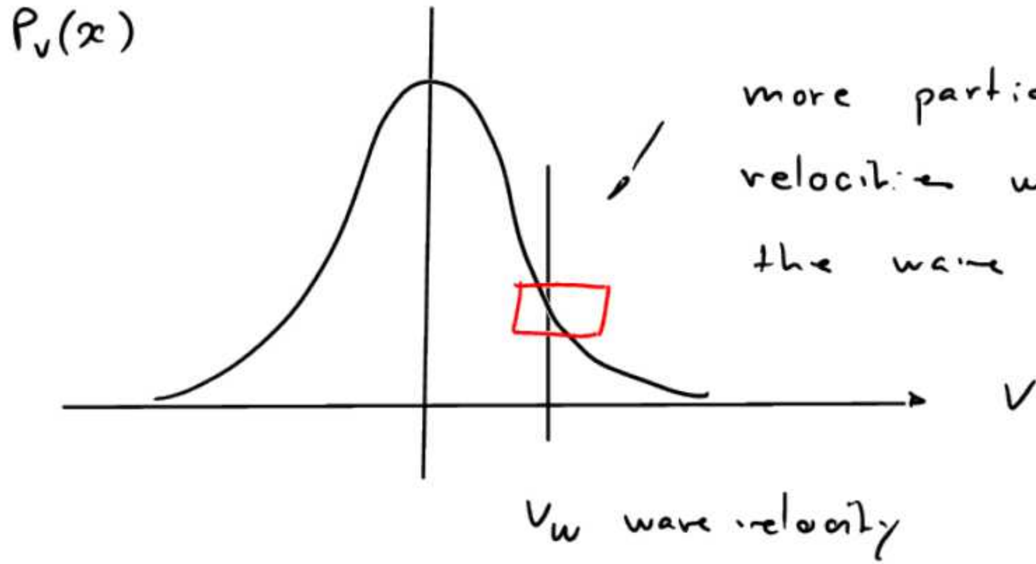
① if a particle with $v > v_w$ is trapped by the wave

$\Rightarrow$ energy is **given** to the wave

② if a particle with $v < v_w$ is trapped by the wave

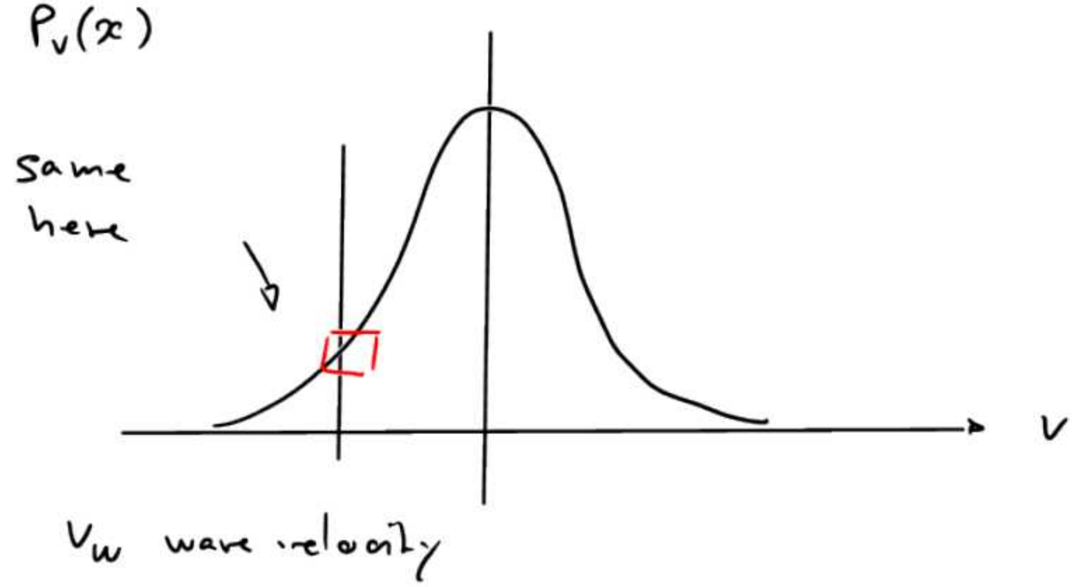
$\Rightarrow$ energy is **taken** from the wave

If the velocity distribution function is <u>Maxwellian</u>:



more particles with lower velocities will interact with the wave

$\boxed{\text{LANDAU DAMPING}}$



The End

Linear differential equations

$$f(x) : \frac{\partial^n f}{\partial x^n} g_n(x) + \dots + \frac{\partial^2 f}{\partial x^2} g_2(x) + \frac{\partial f}{\partial x} g_1(x) + f(x) g_0(x) = c$$

$g_i(x)$: a continuous function

if f_1, f_2 are solutions $a f_1 + b f_2$ is a solution

Illustration

- 1-D continuity equation (ρ, v)

$$\frac{\partial}{\partial t}\rho + \underbrace{\rho \frac{\partial v}{\partial x} + v \frac{\partial \rho}{\partial x}}_{} = 0$$

$\longrightarrow$ those terms mixes ρ and v

- 1-D linearized continuity equation (ρ_{s1}, v_1)

$$\frac{\partial}{\partial t} \rho_{s1} + \rho_0 \frac{\partial v_1}{\partial x} + v_0 \frac{\partial \rho_{s1}}{\partial x} = 0$$

$\longrightarrow$ no mixing