Introduction to Differentiable Manifolds

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**Solutions Series 11 - Alternating tensors, differential forms 2022–12–14**

**Exercise 11.1.** Let $T \in \mathrm{Ten}^k(V)$ be a tensor on a real vector space $V$, and let $T_I = T(E_I)$, defined for any multi-index $I = (i_0, \dots, i_{k-1})$, be the coefficients of $T$ w.r.t. some base $(E_i)_i$ of $V$. Show that $(\sigma T)_I = T_{\sigma^* I}$ for any permutation $\sigma \in S_k$

*Solution.* Denote $J = (j_0, \dots, j_{k-1}) = \sigma^* I = (i_{\sigma(0)}, \dots, i_{\sigma(k-1)})$. Then

$$\begin{aligned}(\sigma T)_I &= (\sigma T)(E_{i_0}, \dots, E_{i_{k-1}}) \\ &= T(E_{i_{\sigma(0)}}, \dots, E_{i_{\sigma(k-1)}}) \\ &= T(E_{j_0}, \dots, E_{j_{k-1}}) \\ &= T_{\sigma^* I}\end{aligned}$$

□

**Exercise 11.2.** Let $M$ be a smooth manifold and let $\omega$ be a differential $k$-form on $M$. We say that $\omega$ is smooth at some point $p \in M$ if the component functions of $\omega$ w.r.t. some chart $\varphi$ (that is defined at $p$) are $\mathcal{C}^r$ at $p$. Show that this does not depend on which chart $\varphi$ we use.

*Solution.* Take two charts $\varphi$, $\widetilde{\varphi}$ that are defined at $p$. Their coordinate vectors are related by the formula $\frac{\partial}{\partial \widetilde{\varphi}^j} = \sum_i \frac{\partial \varphi^i}{\partial \widetilde{\varphi}^j} \frac{\partial}{\partial \varphi^i}$. Using the transformation law for covariant $k$-tensors, we have

$$\widetilde{\omega}_J = \sum_{I=(i_0,\dots,i_{k-1}) \in \underline{n}^k} \frac{\partial \varphi^{i_0}}{\partial \widetilde{\varphi}^{j_0}} \cdots \frac{\partial \varphi^{i_{k-1}}}{\partial \widetilde{\varphi}^{j_{k-1}}} \omega_I.$$

(Note that the sum is over all $k$-indices $I$, not just the increasing indices.)

If the functions $\omega_I$ are $\mathcal{C}^r$ at $p$, since the functions $\frac{\partial \varphi^i}{\partial \widetilde{\varphi}^j}$ are also $\mathcal{C}^r$ (because the transition map $\varphi \circ \widetilde{\varphi}^{-1}$ is $\mathcal{C}^{r+1}$) we conclude that the functions $\widetilde{\omega}_J$ are $\mathcal{C}^r$ at $p$. □

**Exercise 11.3.** The goal of this exercise is to show that the wedge product of alternating covariant tensors in a real vector space $V$ is associative.

(a) If a tensor $T \in \mathrm{Ten}^k V$ is alternating, show that $A(T) = k!\, T$.
(b) For two tensors $S \in \mathrm{Ten}^k V$, $T \in \mathrm{Ten}^\ell V$, show that

$$\begin{aligned}A(A(S) \otimes T) &= k!\, A(S \otimes T) \\ A(S \otimes A(T)) &= \ell!\, A(S \otimes T).\end{aligned}$$

(c) Show that the wedge product of alternating tensors $S \in \mathrm{Alt}^k V$, $T \in \mathrm{Alt}^\ell V$, $R \in \mathrm{Alt}^m V$ is associative:

$$S \wedge (T \wedge R) = S \wedge T \wedge R = (S \wedge T) \wedge R.$$

*Solution.* See Tu's book "An Introduction to Manifolds", Proposition 3.25. □

**Exercise 11.4.** Prove that for each $k$

$$\bigwedge^k T^* M := \coprod_{p \in M} \bigwedge^k T_p^* M$$

are vector bundles over $M$ of rank $\binom{n}{k}$.

***Hint:*** Show that given a coordinate chart $(U, \varphi)$ of $M$ we have a trivialization of $\bigwedge^k T^* M$; given two such trivialization compute the transition function in terms of the change coordinates to conclude.

*Solution.* Let $M$ be a smooth manifold. We consider the set.

$$\bigwedge^k T^*M := \{(p, A) \mid p \in M, A \in \mathrm{Alt}^k(T_pM, \mathbb{R})\}$$

Denote $\pi : \bigwedge^k T^*M \to M$ the projection map $(p, A) \mapsto p$.

To give a smooth manifold structure to the set $\bigwedge^k T^*M$ we will use an atlas of local parametrizations. For each smooth chart $(U, \varphi)$ of $M$ we have a local parametrization $\Phi$ of $\bigwedge^k T^*M$ given by

$$\begin{array}{llll} \Phi : & \varphi(U) \times \mathbb{R}^{\underline{n}^k_\nearrow} & \to & \pi^{-1}(U) \\ & (x, a) & \mapsto & (p = \varphi^{-1}(x), A = \sum_{I=(i_0,\dots,i_{k-1})\in \underline{n}^k_\nearrow} a_I \, \mathrm{d}\varphi^{i_0}|_p \wedge \cdots \wedge \mathrm{d}\varphi^{i_{k-1}}|_p). \end{array}$$

Let $(\widetilde{U}, \widetilde{\varphi})$ be a second chart of $M$ and let $\widetilde{\Phi}$ be the corresponding local parametrization of $\bigwedge^k T^*M$, given by

$$\begin{array}{llll} \widetilde{\Phi} : & \widetilde{\varphi}(\widetilde{U}) \times \mathbb{R}^{\underline{n}^k_\nearrow} & \to & \pi^{-1}(\widetilde{U}) \\ & (\widetilde{x}, \widetilde{a}) & \mapsto & (p = \widetilde{\varphi}^{-1}(\widetilde{x}), A = \sum_{J=(j_0,\dots,j_{k-1})\in \underline{n}^k_\nearrow} \widetilde{a}_J \, \mathrm{d}\widetilde{\varphi}^{j_0}|_p \wedge \cdots \wedge \mathrm{d}\widetilde{\varphi}^{j_{k-1}}|_p). \end{array}$$

The transition map from $\Phi$ to $\widetilde{\Phi}$ is

$$\begin{array}{llll} \widetilde{\Phi}^{-1} \circ \Phi : & \Phi(U \cap \widetilde{U}) \times \mathbb{R}^{\underline{n}^k_\nearrow} & \to & \widetilde{\Phi}(U \cap \widetilde{U}) \times \mathbb{R}^{\underline{n}^k_\nearrow} \\ & (x, a) & \mapsto & (\widetilde{x} = \widetilde{\varphi} \circ \varphi^{-1}(x), \widetilde{a}), \end{array}$$

where

$$\widetilde{a}_J = \sum_{I=(i_0,\dots,i_{k-1})\in \underline{n}^k} \frac{\partial \varphi^{i_0}}{\partial \widetilde{\varphi}^{j_0}} \cdots \frac{\partial \varphi^{i_{k-1}}}{\partial \widetilde{\varphi}^{j_{k-1}}} a_I.$$

(To perform this sum, the function $I \mapsto a_I$, which is initially defined only for increasing $k$-indices, must be extended to an alternating function defined on all $k$-indices.)

These parametrizations $\Phi$ make $\bigwedge^k T^*M$ a smooth manifold by the "one-step smooth manifolds" theorem. To apply this theorem we must verify that all the hypotheses are satisfied. The first 3 hypotheses clearly hold, therefore $\bigwedge^k T^*M$ is a locally Euclidean, second countable topological space with a smooth structure. We must check the last hypothesis, which ensures that $\bigwedge^k T^*M$ is Hausdorff.

Consider two distinct points $(p_i, A^i) \in \bigwedge^k T^*M$, with $i = 0, 1$. Take respective charts $(U_i, \varphi_i)$ such that $p_i \in U_i$ (we choose the same chart if $p_0 = p_1$), and let $(x_i, a^i) = \Phi_i^{-1}(p_i, A^i)$. We have to find sets $W_i \subseteq \mathrm{Dom}(\Phi_i) = \phi_i(U_i) \times \mathbb{R}^{\underline{n}^k_\nearrow}$ such that the two sets $\Phi_i(W_i)$ (for $i = 0, 1$) are disjoint. If the two points $p_i$ are distinct, then we can find respective open neighborhoods $V_i \subseteq U_i$ that are disjoint, and then we set $W_i = \phi_i(V_i) \times \mathbb{R}^{\underline{n}^k_\nearrow}$, so that the sets $\Phi_i(W_i) = \pi^{-1}(V_i)$ are disjoint. Otherwise, if $p_0 = p_1$, then, as said above, we may assume that $\varphi_0 = \varphi_1$. In this case we may obtain disjoint open neighborhoods of the points $(p_i, A^i)$ by applying the open map $\Phi_0$ to two disjoint open neighborhoods of the points $(x_i, a^i)$ in the (Hausdorff!) space $\phi_0(V_0) \times \mathbb{R}^{\underline{n}^k_\nearrow}$. This concludes the proof that $M$ is a smooth manifold with the maps $\Phi$ as smooth local parametrizations.

In addition, each map $\Phi : (x, a) \mapsto (p = \varphi^{-1}(x), A = \sum \dots)$ is linear on the fibers (i.e. if we fix $x$ we get a linear map $a \mapsto A$). This shows that $\bigwedge^k T^*M$ is a vector bundle admitting the maps $\Phi$ as local trivializations. □

**Exercise 11.5 (to hand in ).** For a point $p \in \mathbb{R}^3$ and vectors $v, w \in \mathrm{T}_p\mathbb{R}^3 \equiv \mathbb{R}^3$ we define $\omega|_p(v, w) := \det(p \mid v \mid w)$. Show that $\omega$ is a smooth differential 2-form on $\mathbb{R}^3$, and express $\omega$ as a linear combination of the elementary alternating 2-forms determined by the standard coordinate chart $(x^0, x^1, x^2)$.

**Exercise 11.6** (Some properties of the pullback of differential forms)**.** For $F : M \to N$ a smooth map between smooth manifolds, $\omega \in \Omega^k(N)$, $\beta \in \Omega^\ell(N)$ we have:

(a) $F^*(\alpha \wedge \beta) = F^*(\alpha) \wedge F^*(\beta)$.

*Solution.* Fix $p \in M$ and $X_0, \ldots, X_{k+\ell-1} \in \mathrm{T}_pM$. We have to show that

$$F^*(\alpha_{F(p)} \wedge \beta_{F(p)})(X_0, \ldots, X_{k+\ell-1}) = (F^*\alpha_{F(p)} \wedge F^*\alpha_{F(p)})(X_0, \ldots, X_{k+\ell-1}).$$

From the definition of pullback and wedge product we have

$$\begin{aligned}
&F^*(\alpha_{F(p)} \wedge \beta_{F(p)})(X_0, \ldots, X_{k+\ell-1}) \\
&= \alpha_{F(p)} \wedge \beta_{F(p)} \left(F_*X_0, \ldots, F_*X_{k+\ell-1}\right) \\
&= \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}\sigma\, \alpha_{F(p)}(F_*X_{\sigma(0)}, \ldots, F_*X_{\sigma(k-1)})\, \beta_{F(p)}(F_*X_{\sigma(k)}, \ldots, F_*X_{\sigma(k+\ell-1)}) \\
&= \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}\sigma\, F^*\alpha_{F(p)}(X_{\sigma(0)}, \ldots, X_{\sigma(k-1)})\, F^*\beta_{F(p)}(X_{\sigma(k)}, \ldots, X_{\sigma(k+\ell-1)}) \\
&= (F^*\alpha_{F(p)} \wedge F^*\beta_{F(p)})(X_0, \ldots, X_{k+\ell-1}).
\end{aligned}$$

□

(b) In any coordinate chart $y^i$ on $N$,

$$F^*\left(\sum_{\substack{I=(i_0,\ldots,i_{k-1})\\ 0 \le i_0,\ldots,i_{k-1} < n}} \omega_I \,\mathrm{d}y^I\right) = \sum_{\substack{I=(i_0,\ldots,i_{k-1})\\ 0 \le i_0,\ldots,i_{k-1} < n}} (\omega_I \circ F)\,\mathrm{d}(y^{i_0} \circ F) \wedge \cdots \wedge \mathrm{d}(y^{i_{k-1}} \circ F).$$

*Solution.* Observation: From the definition of the pullback $F^*$ it follows immediately that

$$F^*(\omega + \eta) = F^*\omega + F^*\eta, \quad F^*(f\omega) = (f \circ F)F^*\omega$$

for $\omega, \eta \in \Omega^k(N)$, $f \in C^\infty(N)$. In addition to this we use the following properties of the pullback of 1-forms: let $f \in C^\infty(N)$, and $\omega \in \Omega^1(N)$, then $F^*\,\mathrm{d}f = \mathrm{d}(f \circ F)$ and $F^*(f\sigma) = (f \circ F)F^*\sigma$.

Denoting $I = (i_0, \ldots, i_{k-1})$ an increasing multi-index (i.e. such that $0 \le i_0, \ldots, i_{k-1} < n$), we have

$$\begin{aligned}
F^*\left(\sum_I \omega_I \,\mathrm{d}y^I\right) &= F^*\left(\sum_I \omega_I \,\mathrm{d}y^{i_0} \wedge \cdots \wedge \mathrm{d}y^{i_{k-1}}\right) \\
&= \sum_I (\omega_I \circ F) F^*(\mathrm{d}y^{i_0} \wedge \cdots \wedge \mathrm{d}y^{i_{k-1}}) \\
&= \sum_I (\omega_I \circ F)(F^*\,\mathrm{d}y^{i_0}) \wedge \cdots \wedge (F^*\,\mathrm{d}y^{i_{k-1}}) \\
&= \sum_I (\omega_I \circ F)\,\mathrm{d}(y^{i_0} \circ F) \wedge \cdots \wedge \mathrm{d}(y^{i_{k-1}} \circ F)
\end{aligned}$$

□

(c) $F^*(\omega) \in \Omega^k(M)$.

*Solution.* The last line above is a local expression for $F^*\omega$ defined on the preimage of the domain of the chart $(y^i)$ by $F$. The coefficients $\omega_I \circ F$ are smooth because $\omega_I = \omega_{i_0,\ldots,i_{k-1}}$ are the component functions of a $k$-form and hence smooth. The functions $y^{i_s} \circ F$ are smooth as well and hence their differentials are smooth 1-forms. Now the only missing ingredient is that the wedge product of a smooth differential forms is a smooth differential form. But this is clear because the component functions of the wedge product are sums of products of component functions of the original forms and hence smooth. □

**Exercise 11.7.** Define a 2-form $\omega$ on $\mathbb{R}^3$ by

$$\omega = x\ \mathrm{d}y \wedge \mathrm{d}z + y\ \mathrm{d}z \wedge \mathrm{d}x + z\ \mathrm{d}x \wedge \mathrm{d}y.$$

(a) Compute $\omega$ in spherical coordinates.

*Solution.* This is the same 2-form of a previous exercise, $\omega|_p(v, w) = \det(p, v, w)$. We want to express $\omega$ in polar coordinates, i.e. compute $\Phi^*\omega$ where

$$\Phi(r, \theta, \varphi) = (r \sin\theta \cos\varphi r \sin\theta \sin\varphi, r \cos\theta).$$

We can write $\Phi^*\omega = a\ \mathrm{d}r \wedge \mathrm{d}\varphi + b\ \mathrm{d}r \wedge \mathrm{d}\theta + c\ \mathrm{d}\varphi \wedge \mathrm{d}\theta$, where $a, b, c$ are scalar functions of $(r, \varphi, \theta)$ obtained as follows. Denoting $p = \Phi(r, \theta, \varphi)$, we have

$$a = \omega_p \left( \frac{\partial\Phi}{\partial r}, \frac{\partial\Phi}{\partial\theta} \right) = \det \left( p, \frac{\partial\Phi}{\partial r}, \frac{\partial\Phi}{\partial\theta} \right) = 0$$

because $\frac{\partial\Phi}{\partial p} = \frac{1}{r}p$ is parallel to $p$. For the same reason we have

$$b = \omega_p \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial\varphi} \right) = \det \left( p, \frac{\partial\Phi}{\partial r}, \frac{\partial\Phi}{\partial\varphi} \right) = 0.$$

The remaining function is

$$c = \omega_p \left( \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi} \right) = \det \left( p, \frac{\partial\Phi}{\partial\theta}, \frac{\partial\Phi}{\partial\varphi} \right) = \dots$$

Now we note that the matrix $\left( p, \frac{\partial\Phi}{\partial\theta}, \frac{\partial\Phi}{\partial\varphi} \right)$ is very similar, differing only by a factor $r$ on the first column, to the Jacobian matrix of $\Phi$,

$$J = \left( \frac{\partial\Phi}{\partial r}, \frac{\partial\Phi}{\partial\theta}, \frac{\partial\Phi}{\partial\varphi} \right) = \begin{pmatrix} \sin\theta \cos\varphi & r \cos\theta \cos\varphi & -r \sin\theta \sin\varphi \\ \sin\theta \sin\varphi & r \cos\theta \sin\varphi & r \sin\theta \cos\varphi \\ \cos\theta & -r \sin\theta & 0 \end{pmatrix}$$

whose determinant is known to be $\det J = r^2 \sin\theta$. Thus we can easily compute the determinant that we are interested in,

$$c = \dots = r \det J = r^3 \sin\theta.$$

We conclude that $\Phi^*\omega = r^3 \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\varphi$.

□

(b) Show that $\omega|_{\mathbb{S}^2}$ is nowhere 0.

*Solution.* We know that $\omega|_p \neq 0$ iff $p \neq 0$ because $\omega|_p(v, w) = \det(p, v, w)$. □

(c) Compute the exterior derivative $\mathrm{d}\omega$.

*Solution.*

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x + \mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y \\ &= 3\ \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z. \end{aligned}$$

□