MCAA lecture 13

Coupling from the past (exact simulation)

Aim: to sample from π exactly.

Last time: forward coupling does not work

Today: backward coupling
(aka Propp-Wilson algorithm)

Reminder: π is the stat. dist. of a MC X

The algorithm

(iid uniform)

1. Draw random numbers $U_0, U_{-1}, U_{-2} \dots$ once and for all!

2. Set $T_0 = -1$.

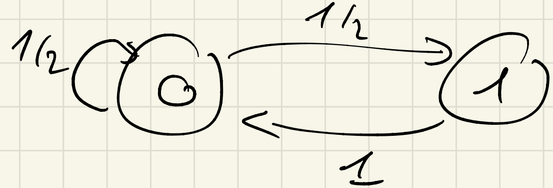
3. Run the MC X from time T_0 until time 0, using the above random numbers:

$$X_{n+1}^{i,T_0} = \phi\left(X_n^{i,T_0}, U_{n+1}\right), \quad X_{T_0}^{i,T_0} = i$$

and do this for all possible initial states $i \in S$.

4. If $X_0^{i,T_0} = X_0^{j,T_0}$ $\forall i \neq j \in S$, output this value of X_0 (= the sample); otherwise, set $T_0 \leftarrow T_0 - 1$ and restart in 3.

<u>Note</u>: In practice, in step 4, go for $T_0 \leftarrow 2T_0$ instead of $T_0 \leftarrow T_0 - 1$.

Example : 

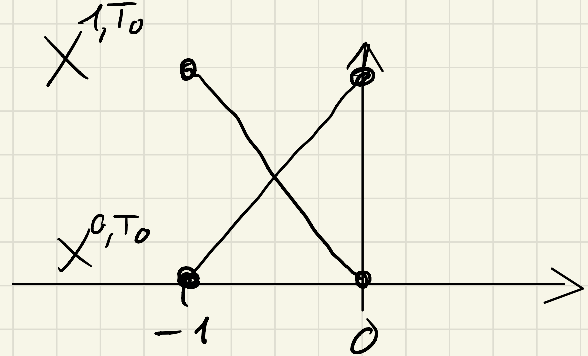
$\pi = \left(\frac{2}{3}, \frac{1}{3}\right)$

$$X_{n+1} = \phi(X_n, U_{n+1}) \quad \text{where} \begin{cases} \phi(0,u) = 1_{u \le \frac{1}{2}} \\ \phi(1,u) = 0 \end{cases}$$

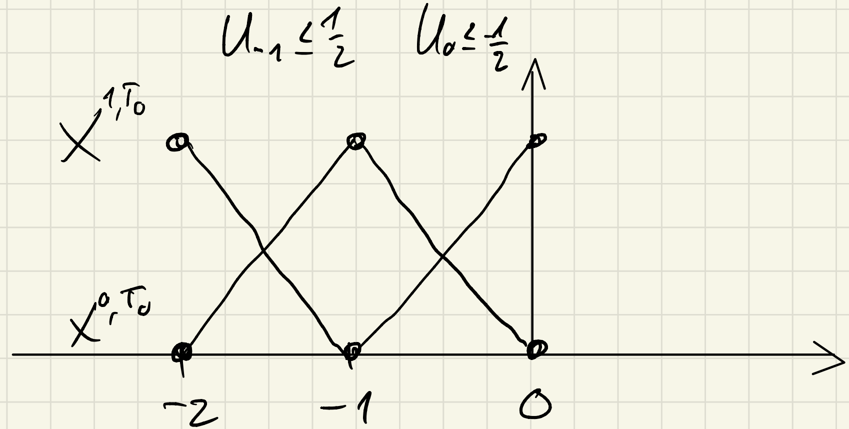
here: U_0, U_{-1}, U_{-2} iid $\sim \mathcal{U}[0,1]$

for ex: $U_0 \le \frac{1}{2}$, $U_{-1} \le \frac{1}{2}$, $\underline{U_{-2} > \frac{1}{2}}$, $U_{-3} \le \frac{1}{2}$ ...

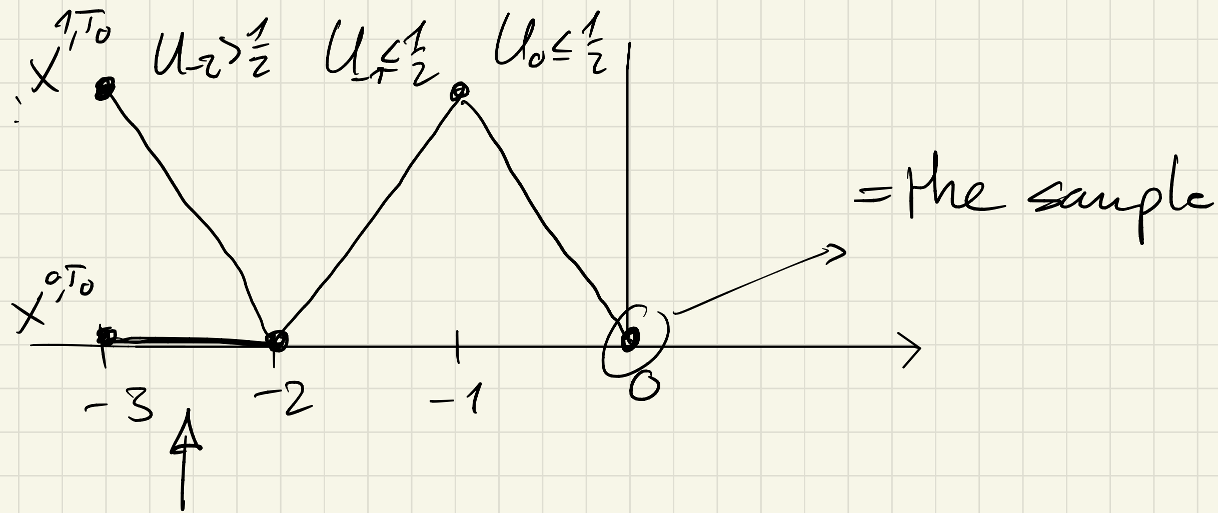
$T_0 = -1$:



$\underline{T_0 = -2}$:



$\underline{T_0 = -3}$:



$$\mathbb{P}(\text{output state} = 0)$$

$$= \mathbb{P}\Big(U_0 > \tfrac{1}{2} \text{ or } U_0 \le \tfrac{1}{2}, U_{-1} \le \tfrac{1}{2}, U_{-2} > \tfrac{1}{2} \text{ or }$$

$$U_0 \le \tfrac{1}{2}, U_{-1} \le \tfrac{1}{2}, U_{-2} \le \tfrac{1}{2}, U_{-3} \le \tfrac{1}{2}, U_{-4} > \tfrac{1}{2} \text{ or} \dots \Big)$$

$$= \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \dots = \frac{1}{2}\left(1 + \frac{1}{4} + \frac{1}{16} + \dots\right)$$

$$= \frac{1}{2} \cdot \sum_{k \ge 0} \frac{1}{4^k} = \frac{1}{2}\left(\frac{1}{1 - \frac{1}{4}}\right) = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3} \checkmark$$

$$\mathbb{P}(\text{output state } 1)$$

$$= \mathbb{P}\Big(U_0 \le \tfrac{1}{2},\ U_{-1} > \tfrac{1}{2} \ \text{ or } \ U_0 \le \tfrac{1}{2}, U_{-1} \le \tfrac{1}{2},\ U_{-2} \le \tfrac{1}{2}, U_{-3} > \tfrac{1}{2}$$

$$\text{or } \ldots \Big)$$

$$= \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots = \sum_{k \ge 1} \frac{1}{4^k} = \frac{1}{1 - \frac{1}{4}} - 1$$

$$= \frac{4}{3} - 1 = \frac{1}{3} \quad \checkmark$$

$2T_0$
T_0
0

__Theorem__ (Propp-Wilson)

If $\exists L > 0$, s.t. $\mathbb{P}(X_0^{i,-L} = X_0^{j,-L} \ \forall i \neq j) > 0$

Then the Propp-Wilson algorithm outputs in __finite time__ a sample X_0 which is distributed according to π.

Proof

- Assumption means:

$\exists L > 0$ s.t. $\mathbb{P}($coalescence takes place in L steps$)$
$= p > 0$

So:

$\mathbb{P}($coalescence does not take place in L steps$) = 1-p < 1$

So: $\mathbb{P}($coalescence does not take place in $k \cdot L$ steps$) \leq (1-p)^k \xrightarrow[k\to\infty]{} 0$

So: $\mathbb{P}($coalescence takes place in finite time$) = 1$

- Let $L_0 = \inf\{L > 0 : X_0^{i,-L} = X_0^{j,-L} = X_0 \quad \forall i \neq j \in S\}$

Assume $X_0 \sim \mu$. Claim: $\mu = \pi$

- $X_0^{i,-L_0} \sim \mu$, so $X_1^{i,-L_0} \sim \mu \cdot P$

- time-homogeneity: $X_1^{i,-L_0} \sim X_0^{i,-L_0-1} = X_0 \sim \mu$

- so $\mu = \mu \cdot P$ so $\mu = \pi$
(by uniqueness)

$\#$

Problem: to check $X_0^{i,T_0} = X_0^{j,T_0} \quad \forall i \neq j \in S$

is too costly in practice

Solution: extra assumptions:

- S has a partial ordering $\preceq$

  (ex: $x, y \in \{0,1\}^d$: $x \preceq y$ if $x_i \leq y_i \; \forall i$)

- $\exists\, \underline{x}$ and $\overline{x}$ in S st. $\underline{x} \preceq x \preceq \overline{x} \quad \forall x \in S$

  (ex: $\underline{x} = (0, \dots, 0)$, $\overline{x} = (1, \dots, 1)$)

- There exists a <u>monotone</u> random mapping representation Φ of the MC X, i.e.:

  if $x \leq y$, then $\Phi(x,u) \leq \Phi(y,u)$ $\forall u$

<u>Example</u>: Gibbs sampling for the ferromagnetic Ising model

state $\sigma \longrightarrow \tau_{\pm}$ where $\begin{cases} (\tau_{\pm})_w = \sigma_w & \forall w \neq v \\ (\tau_{\pm})_v = \pm 1 \end{cases}$

(pick $v \in V$ at random)

and $\mathbb{P}(\sigma \to \tau_\pm) = \frac{1}{2}\left(1 \pm \tanh\left(\beta h_v^{(loc)}\right)\right)$

where $h_v^{(loc)} = \sum_{\substack{w \in V: \\ (v,w) \in E}} J_{vw} \sigma_w + h_v$

<u>Monotone random mapping representation:</u>

$S = \{-1,+1\}^V$: $\sigma \preceq \tau$ if $\sigma_v \leq \tau_v$ $\forall v \in V$

$\underline{\sigma} = (-1, \dots, -1)$, $\overline{\sigma} = (+1, \dots, +1)$

$$\big(\Phi(\sigma, u)\big)_w = \begin{cases} \sigma_w & \forall w \neq v \\ +1 & \text{if } 0 \leq u \leq \frac{1}{2}\left(1+\tanh\left(\beta h_v^{(loc)}\right)\right) \\ -1 & \text{if } \frac{1}{2}\left(1+\tanh\left(\beta h_v^{(loc)}\right)\right) < u \leq 1 \end{cases}$$

__Claim:__ if $J_{vw} \geq 0$ (ferromagnetic model) then Φ is monotone (so it suffices to check that $X_0^{\underline{\sigma}, T_0} = X_0^{\overline{\sigma}, T_0}$ to output a sample)

Proof of the claim:

$\forall u$

if $\sigma \preceq \sigma'$, then $\phi(\sigma, u) \preceq \phi(\sigma', u)$

Indeed: $h_v^{(loc)} = \sum_{\substack{w \in V: \\ (v,w) \in E}} \underbrace{J_{vw}}_{\geq 0} \sigma_w + h_v$

so if $\underbrace{\sigma \preceq \sigma'}$, then $h_v^{(loc)} \leq (h_v^{(loc)})'$

$\Longrightarrow \sigma_v \leq \sigma_v'$ & $h_v^{(loc)} \leq (h_v^{(loc)})'$

$\Longrightarrow (\phi(\sigma, u))_v \leq (\phi(\sigma', u))_v$ #