Stability of collisionless systems

3rd part

Outlines

The stability of uniformly rotating systems

The stability of rotating disks : spiral structures

- Spirals properties
- The dispersion relation for a razor thin fluid disk

The origin of spiral structures: another view

Vertical instabilities : Nature is always more tricky…

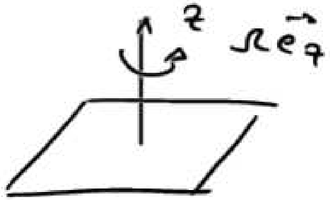
The stability of uniformly rotating systems

The stability of uniformely rotating systems

- Flattened systems : the geometry is more complex
- Reservoir of kinetic energy (rotation) to feed unstable modes

The uniformly rotating sheet

- infinite disk of zero thickness with surface density Σ_0
  - plane $z = 0$
  - rotation $\widehat{\Omega} = \Omega \vec{e}_z$



- 2D perturbation / evolution (no warp, no bending)

Equations in the rotating frame

$\Sigma_d(x,y)$, $\phi_d(x,y)$, $P(x,y)$, $\vec{v}(x,y)$

5 unknown
5 equations

① Continuity equation

$$\boxed{\frac{\partial \Sigma}{\partial t} + \vec{\nabla}(\Sigma \vec{v}) = 0}$$

② Euler Equation

$$\boxed{\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = - \frac{\vec{\nabla} p}{\Sigma_d} - \vec{\nabla}\phi \overbrace{- 2\Omega(v_x \vec{e_x} + v_y \vec{e_y})}^{\text{Coriolis force}} + \overbrace{\Omega^2(x \vec{e}_x + y \vec{e_y})}^{\text{Centrifugal force}}}$$

③ Poisson equation

$$\boxed{\nabla^2 \phi = 4\pi G \Sigma \delta(z)}$$

④ Equation of state (barotropic)

$$\boxed{p(x,y) = p(\Sigma(x,y))}$$

The response of the system to a weak perturbation $-\varepsilon \vec{\nabla}\phi_e$

$$\Sigma_{d0} \rightarrow \Sigma_{d0} + \varepsilon \Sigma_{d1}(x,y,t)$$

$$\vec{v}_0 = 0 \rightarrow \varepsilon V_1(x,y,t)$$

$$\Sigma_0 \rightarrow \Sigma_0 + \varepsilon \Sigma_1(x,y,t)$$ total density

$$\phi_0 \rightarrow \phi_0 + \varepsilon \phi_1(x,y,t)$$ total potential

$$= \varepsilon \phi_{d1}(x,y,t) + \varepsilon \phi_e(x,y,t)$$

A first order in ε, we get

① Continuity equation
$$\frac{\partial}{\partial t}\Sigma_{d1} + \Sigma_0 \vec{\nabla}\vec{V}_1 = 0$$

② Euler Equation
$$\frac{\partial}{\partial t}\vec{V}_1 = -\frac{v_s^2}{\Sigma_0}\vec{\nabla}\Sigma_{d1} - \vec{\nabla}\phi_1 - \underbrace{2\vec{\Omega}\times\vec{V}_1}_{\text{contribution of the rotation}}$$

③ Poisson equation
$$\nabla^2\phi_1 = 4\pi G \Sigma_1 \delta(z)$$

④ "Equation of state"
$$v_s^2 = \left.\frac{\partial p}{\partial \Sigma}\right|_{\Sigma_0}$$

the rest is similar to the homogeneous case

Solutions of the form

$$\Sigma_{d1}(x,y,t) \sim e^{i(\vec{k}\vec{x} - \omega t)}$$

leads to : (without the perturbation)

$$\boxed{\omega^2 = v_s^2 k^2 - 2\pi G|k|\Sigma_0 + \underbrace{4\Omega^2}_{\text{the rotation helps to stabilize}}}$$

(Dispersion relation for an homogeneous fluid

$$\boxed{\omega^2 = v_s^2 k^2 - 4\pi G\rho_0}$$

)

Dispersion relation for a uniformely rotating stellar sheet

- if the velocity DF is Maxwellian

$$f_0\Big(\underset{\substack{\uparrow\\ \text{in the}\\ \text{plane}}}{\vec{v}}\Big) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}}\, e^{-\frac{v^2}{2\sigma^2}}$$

$$\boxed{\omega^2 = \sigma^2 k^2 - 2\pi G|k|\Sigma_0 + 4\Omega^2}$$

Stability of collisionless systems

The stability of rotating disks:

spiral structures

Spiral galaxies : disky structures

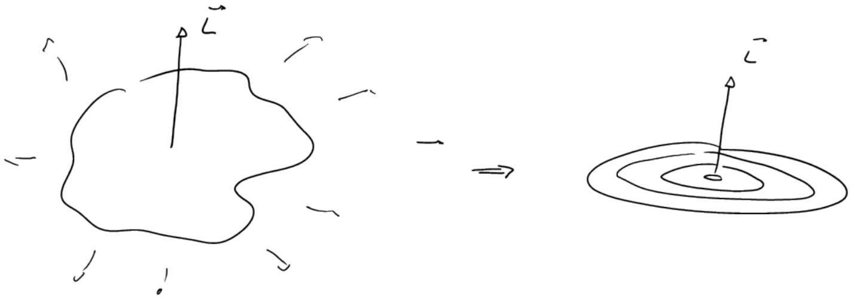
M83

NGC4945

Questions:

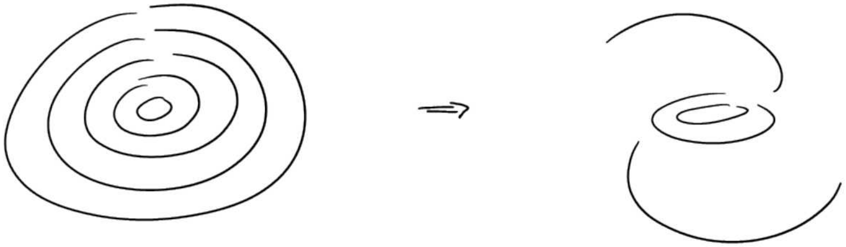
Why galaxies are disks ?

- Gas radiates energy but not angular momentum.
- Circular orbits have a minimum energy for a given angular momentum.
- For a given angular momentum distribution, the state of the lowest energy is a flat disk.



Why disk galaxies display complex structures : bars, spirals ?

- Dynamically cool systems (low velocity dispersion) are strongly unstable
- Further cooling requires to avoid the constraints provided by the angular momentum conservation : <u>need to break the symmetry</u> !

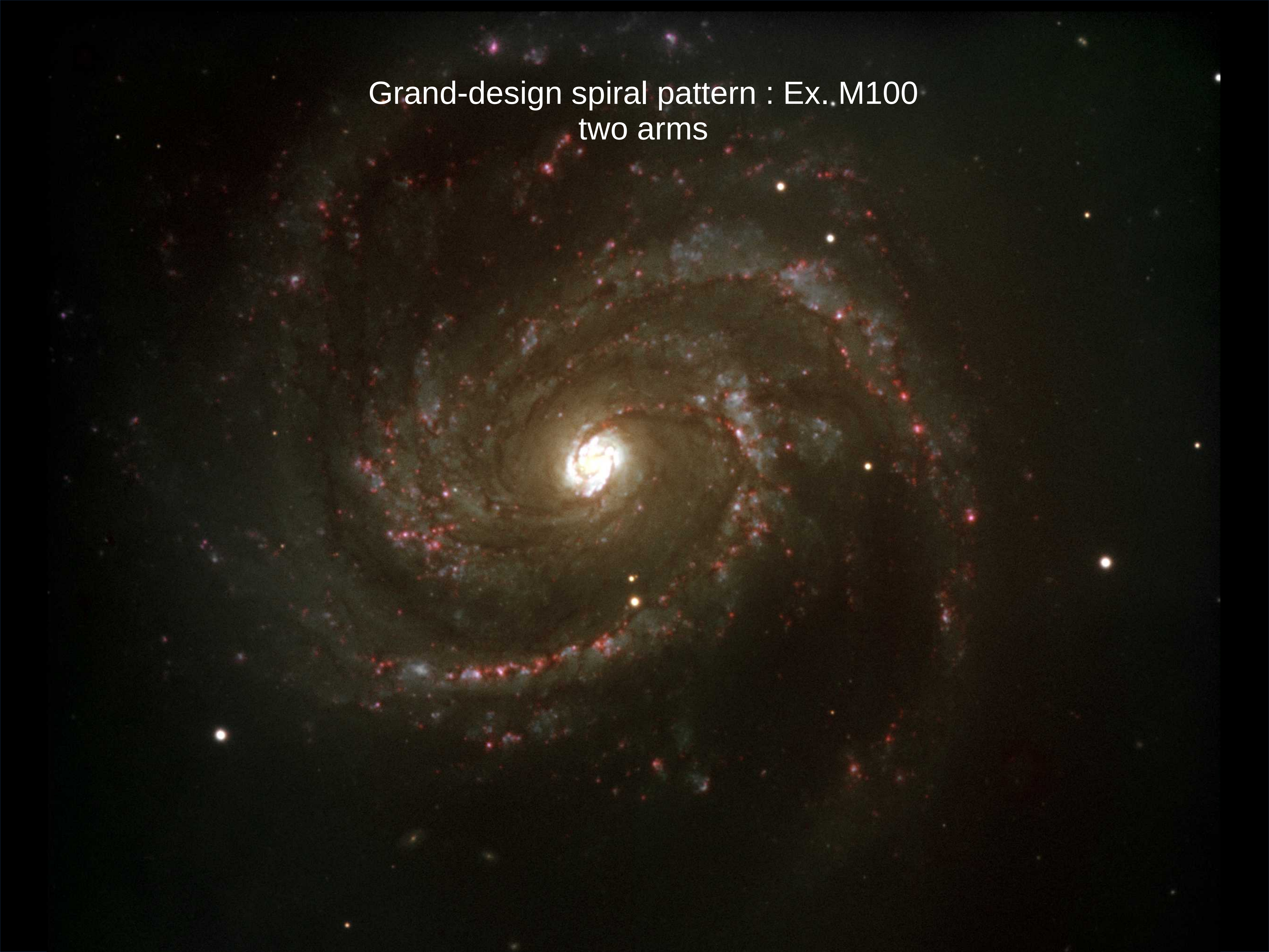


Properties of spirals:

Different spiral patterns:

- Grand-design
- Intermediate-scale
- Flocculent

Grand-design spiral pattern : Ex. M100
two arms



Intermediate-scale spiral pattern : Ex. M101

Flocculent spirals : Ex. M63

Properties of spirals:

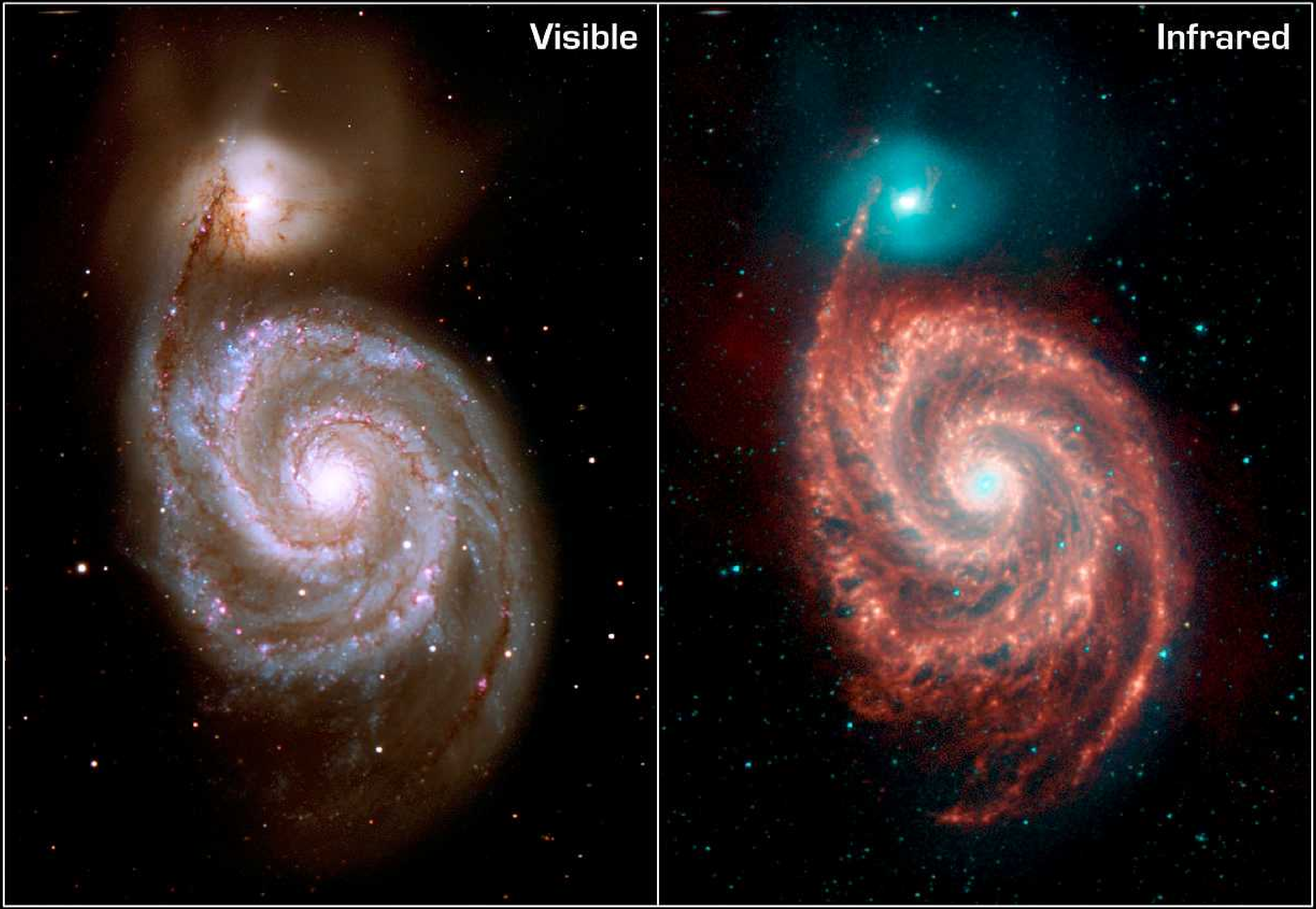
Different spiral patterns:

- Grand-design
- Intermediate-scale
- Flocculent

The bulk of the matter participates to the spirals

- Coherence in different wavelengths tracing different components

Grand-design spiral pattern

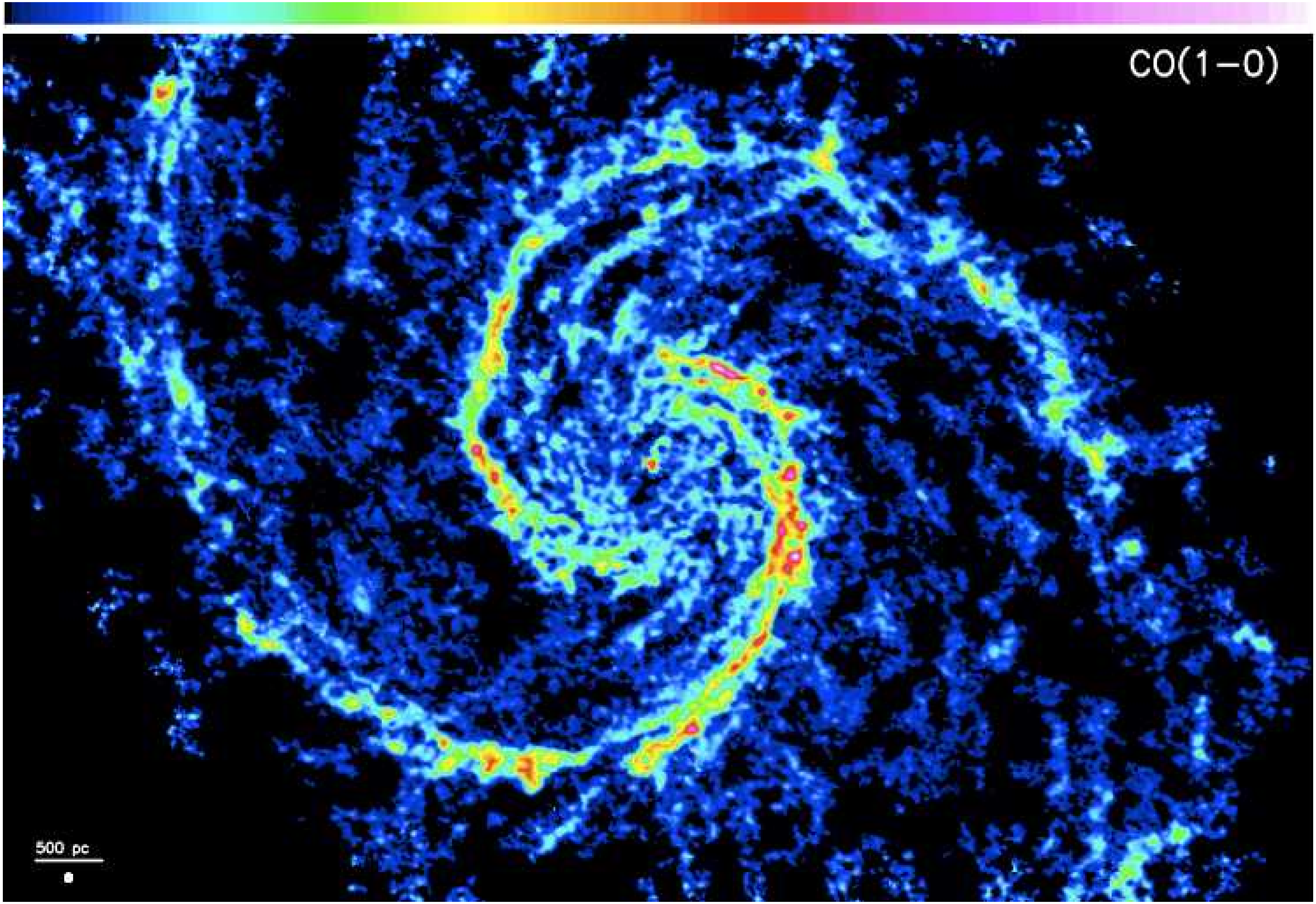


Spiral Galaxy M51 ("Whirlpool Galaxy")

Spitzer Space Telescope • IRAC

NASA / JPL-Caltech / R. Kennicutt (Univ. of Arizona)

ssc2004-19a



Schinnerer et al. 2013

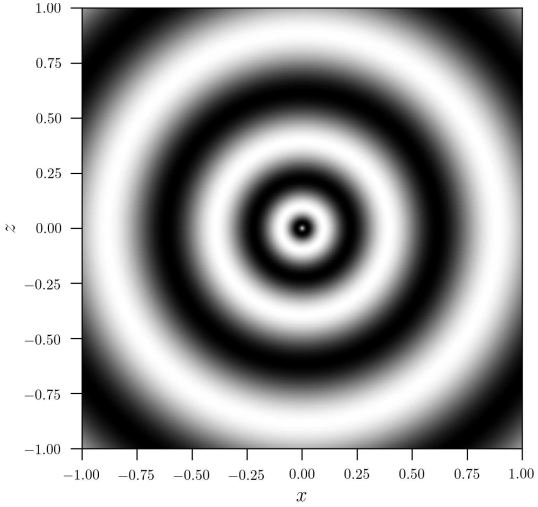
Model spiral arms

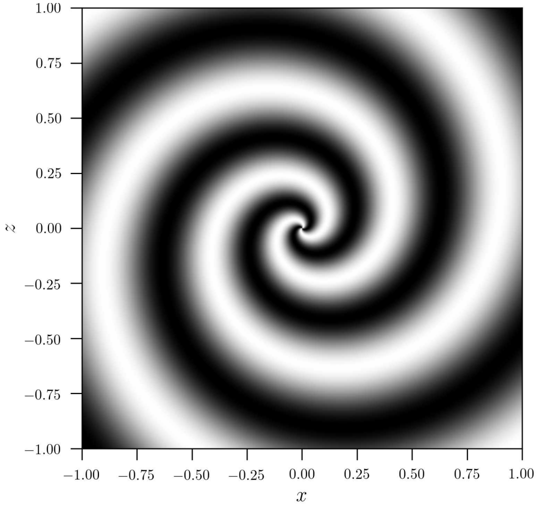
Surface density of a spiral galaxy

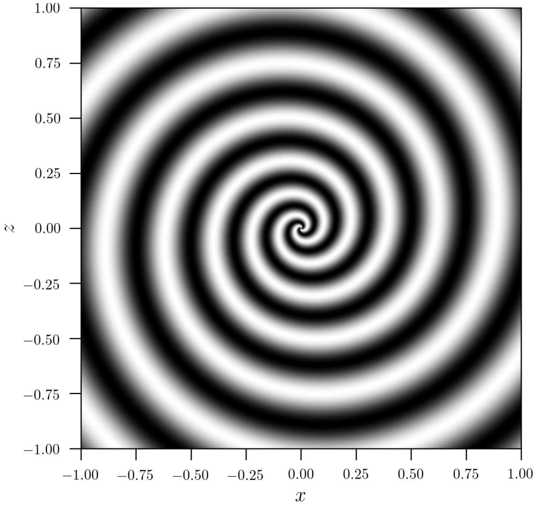
shape function

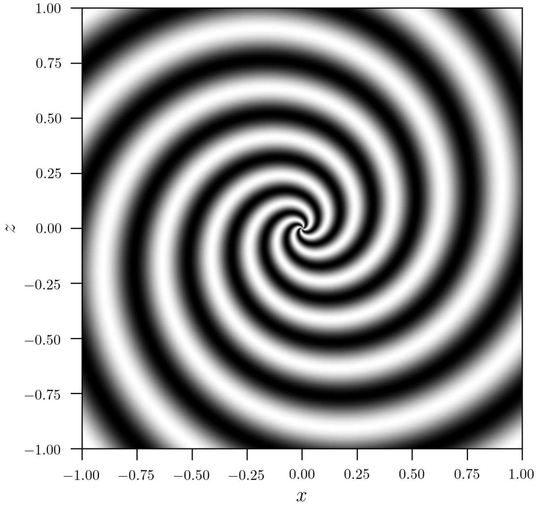
$$\Sigma(R,\phi,t) = Re\left[H(R,t)e^{i(m\phi+f(R,t)-\omega t)}\right]$$

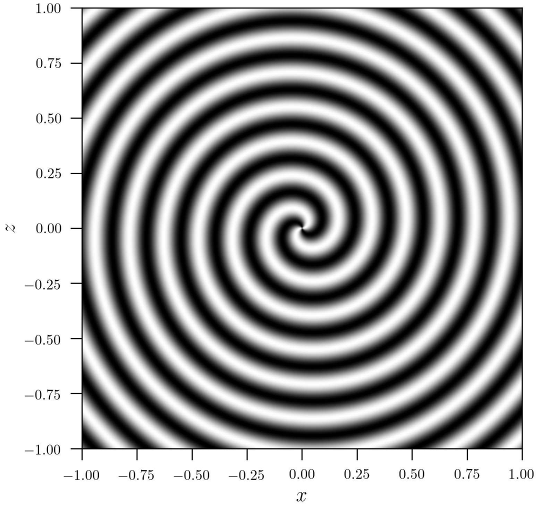
arms

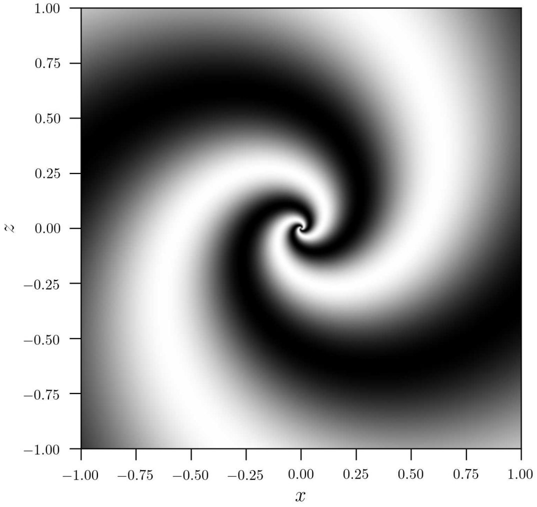

$m = 0, f = 20R^{1/2}$


$m = 2, f = 20R^{1/2}$

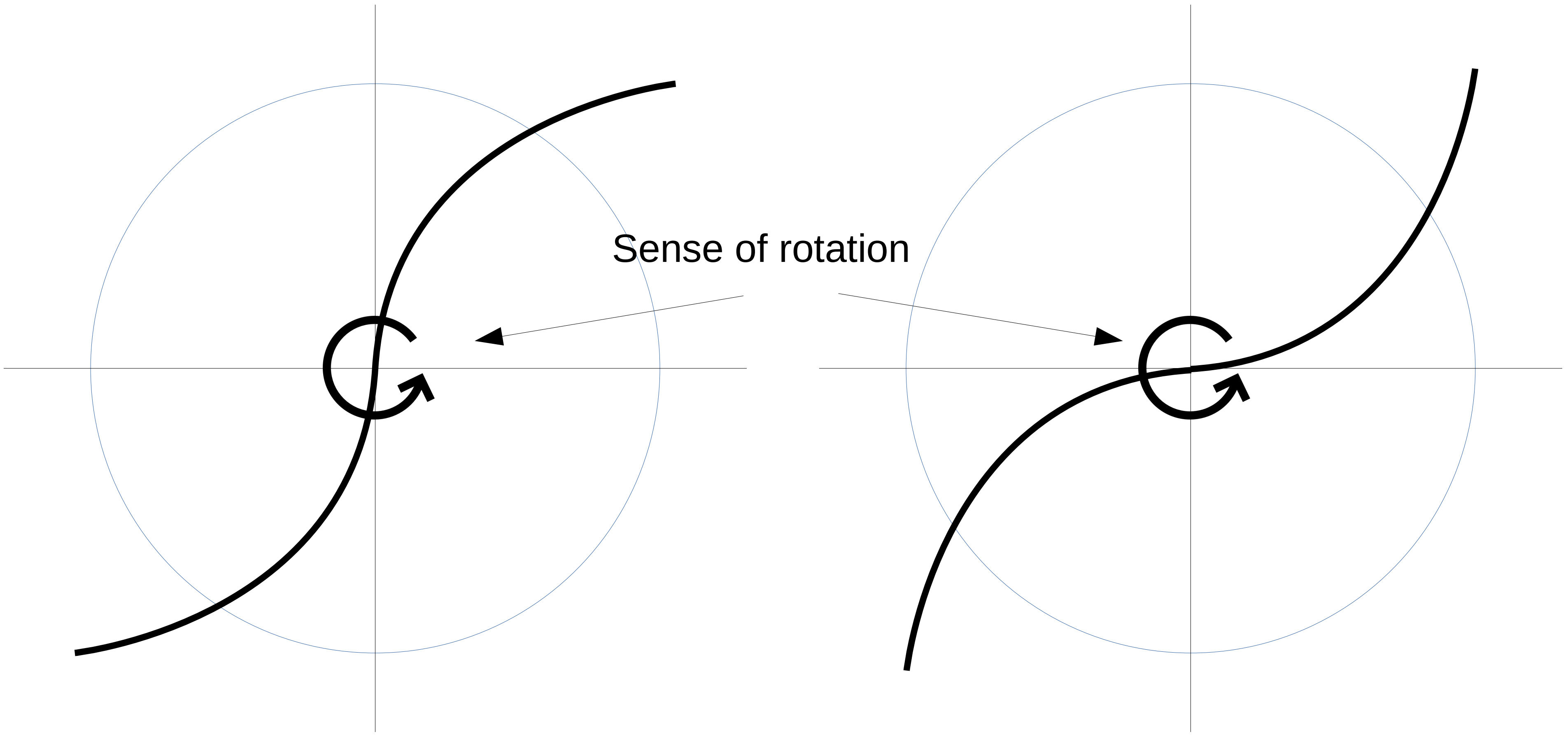

$m = 2, f = 40R^{1/2}$


$m = 4, f = 40R^{1/2}$


$m = 2, f = 40R$


$m = 2, f = 40R^{0.1}$

Leading and trailing spirals:



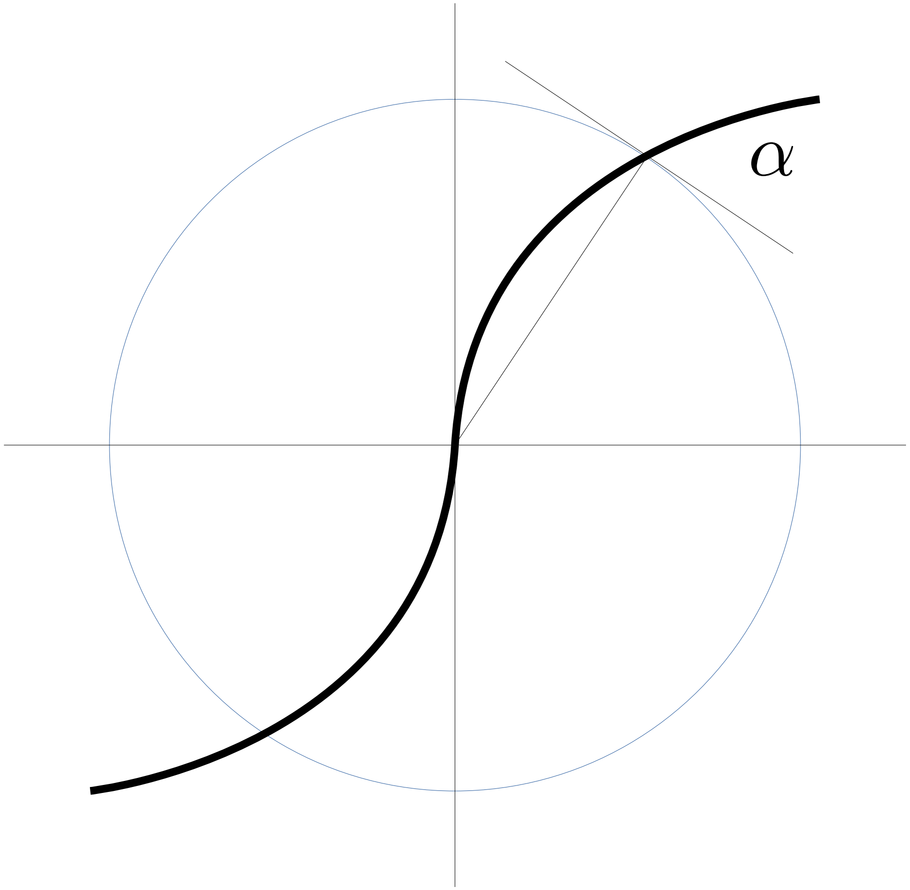
trailing arms

leading arms

The majority of spiral arms are trailing !

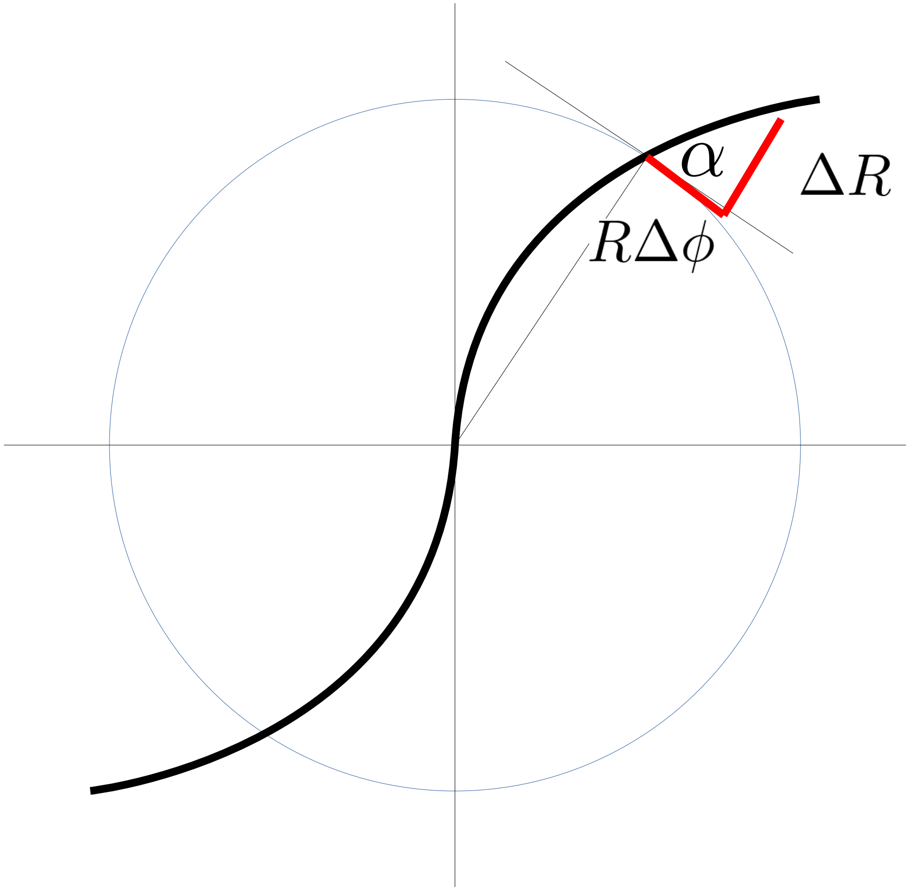
Pitch angle

Definition: pitch angle : angle between the tangent of a circle and the spiral



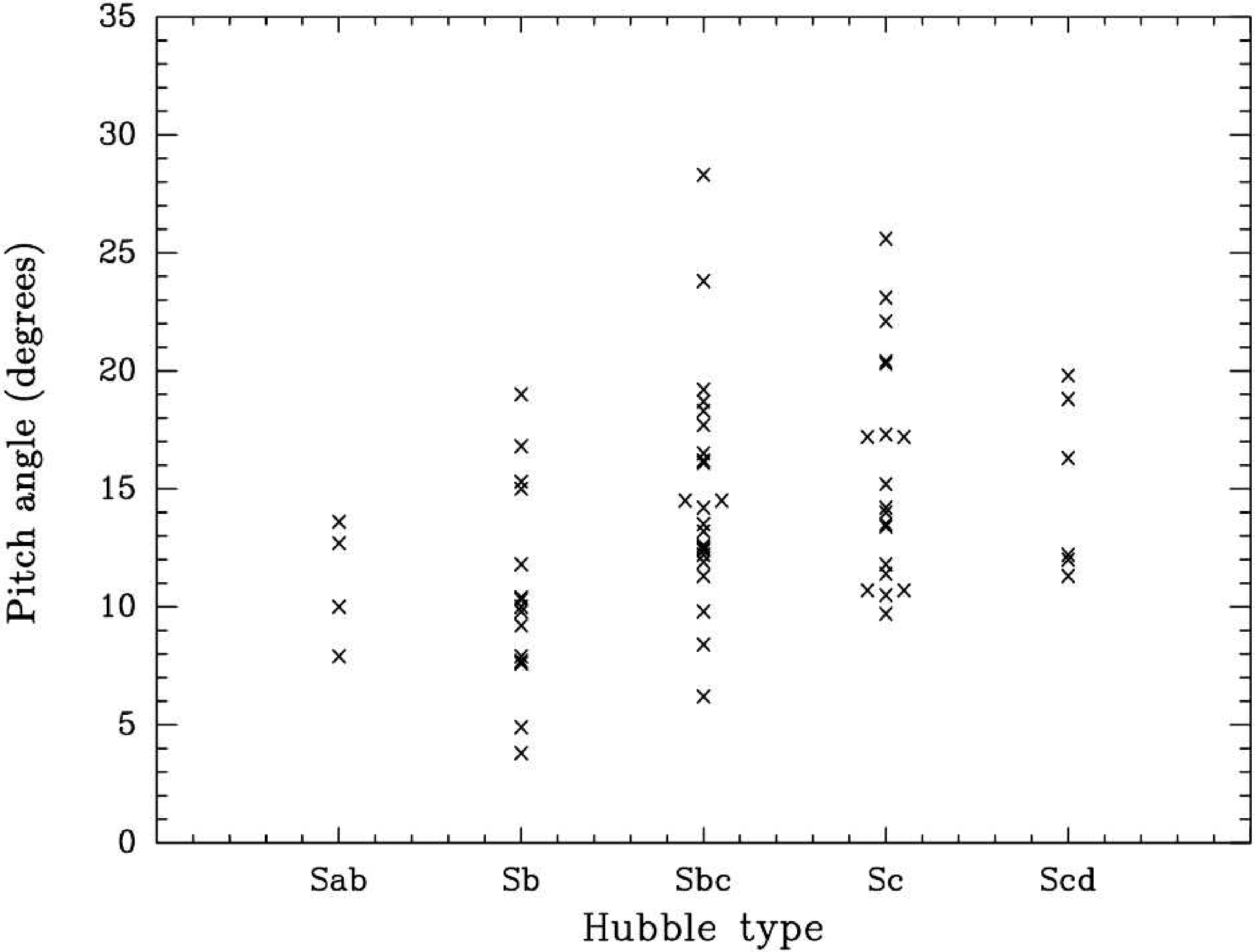
Pitch angle

Definition: pitch angle : angle between the tangent of a circle and the spiral



$$\tan\alpha = \frac{\Delta R}{R\Delta\phi}$$

Pitch angle as a function of the Hubble type (Ma 2002)



Stability of collisionless systems

Origin of the spiral structure:

differential rotation ?

Winding problem

Are observed spirals the result of differential rotation ?

Assume a constant velocity curve

$$V(R) = R\,\Omega(R) \;\hat{=}\; 200\,\mathrm{km/s} = V_0$$

$$\Omega(R) = \frac{V_0}{R}$$

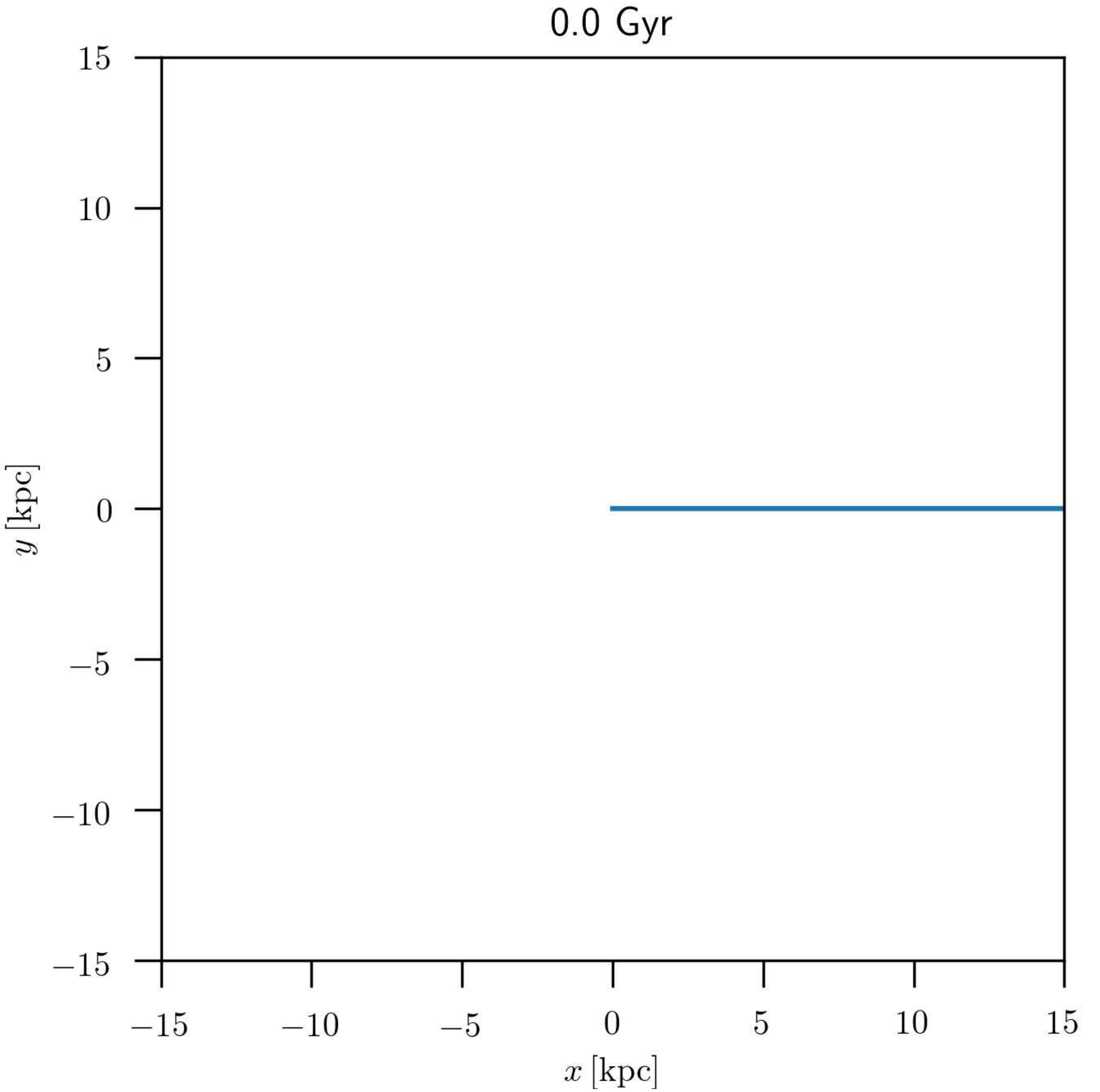
$$\phi(R,t) = \Omega(R)\,t + \phi_0 = \frac{V_0}{R}\,t + \phi_0$$

$$\frac{\partial\phi}{\partial R} = -\frac{V_0 t}{R^2} \Longrightarrow \frac{\Delta R}{R\,\Delta\phi} = \frac{R}{t V_0} \Longrightarrow \alpha = \arctan\left(\frac{R}{t V_0}\right)$$

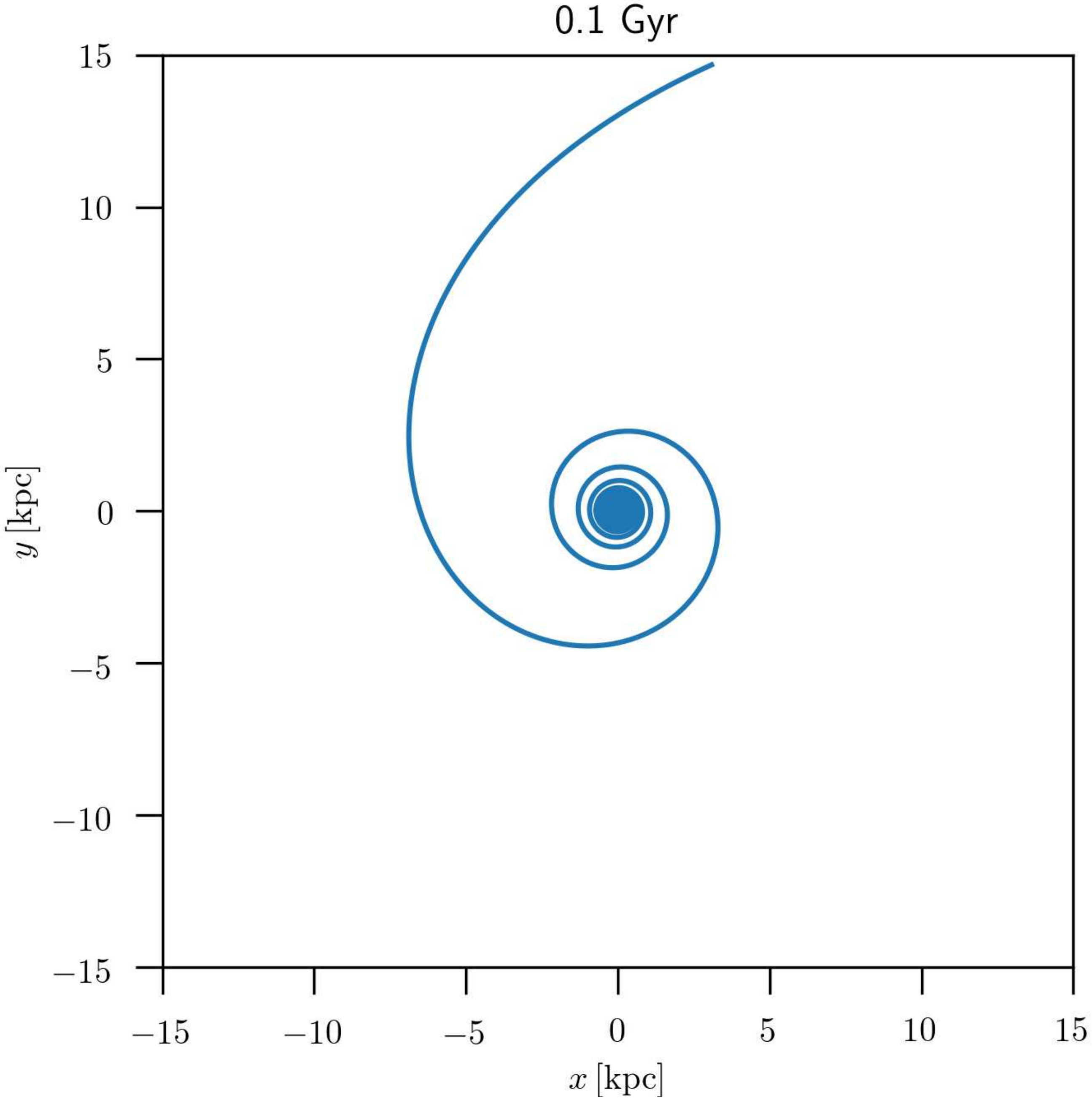
for $t = 10\,\mathrm{Gyr}$, $R = 5\,\mathrm{kpc}$ $V_0 = 200\,\mathrm{km/s}$

$$\boxed{\alpha = 0.14^\circ} \ll 10^\circ - 15^\circ \quad ⚠$$

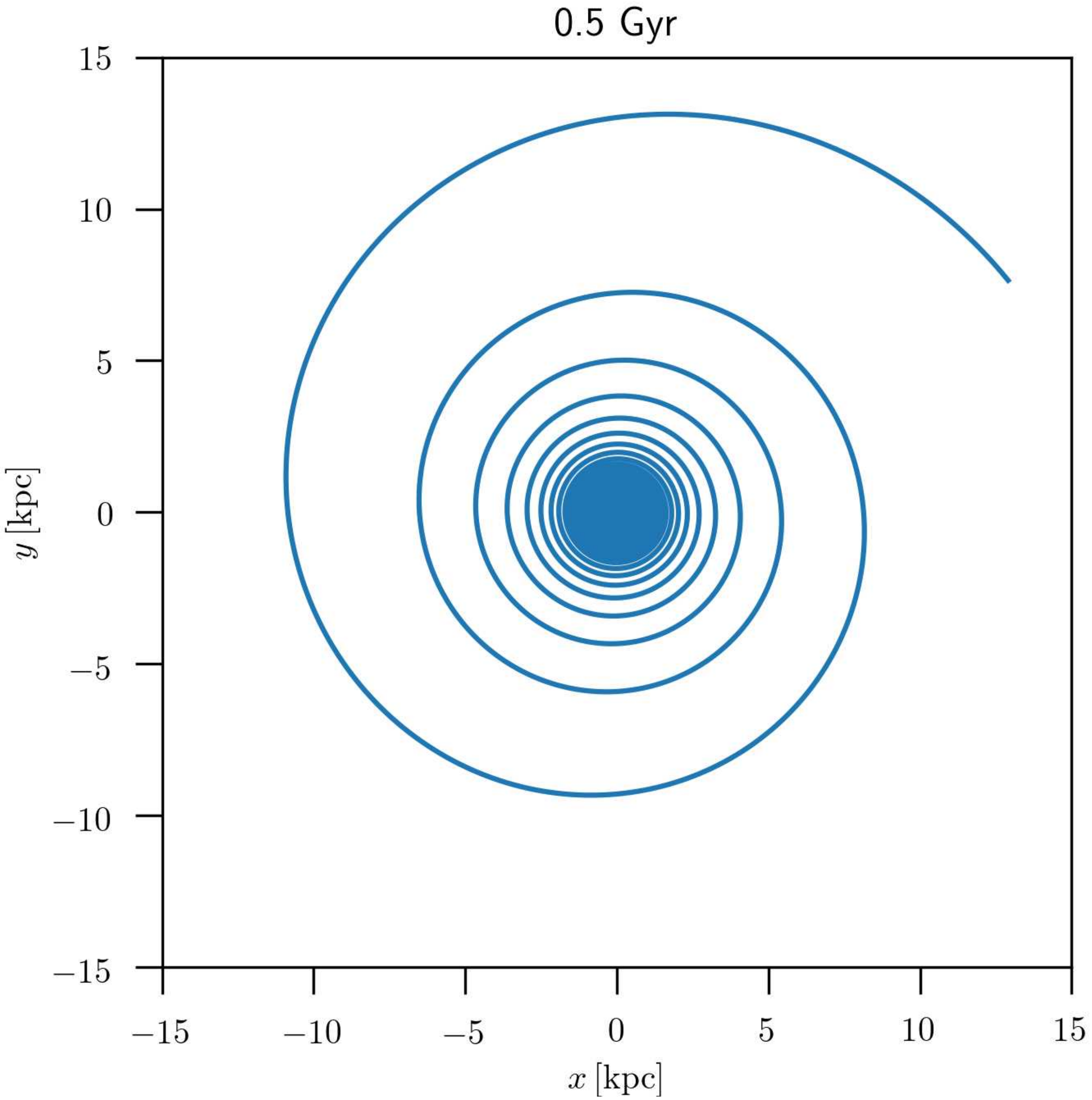
Winding problem : V=200 km/s



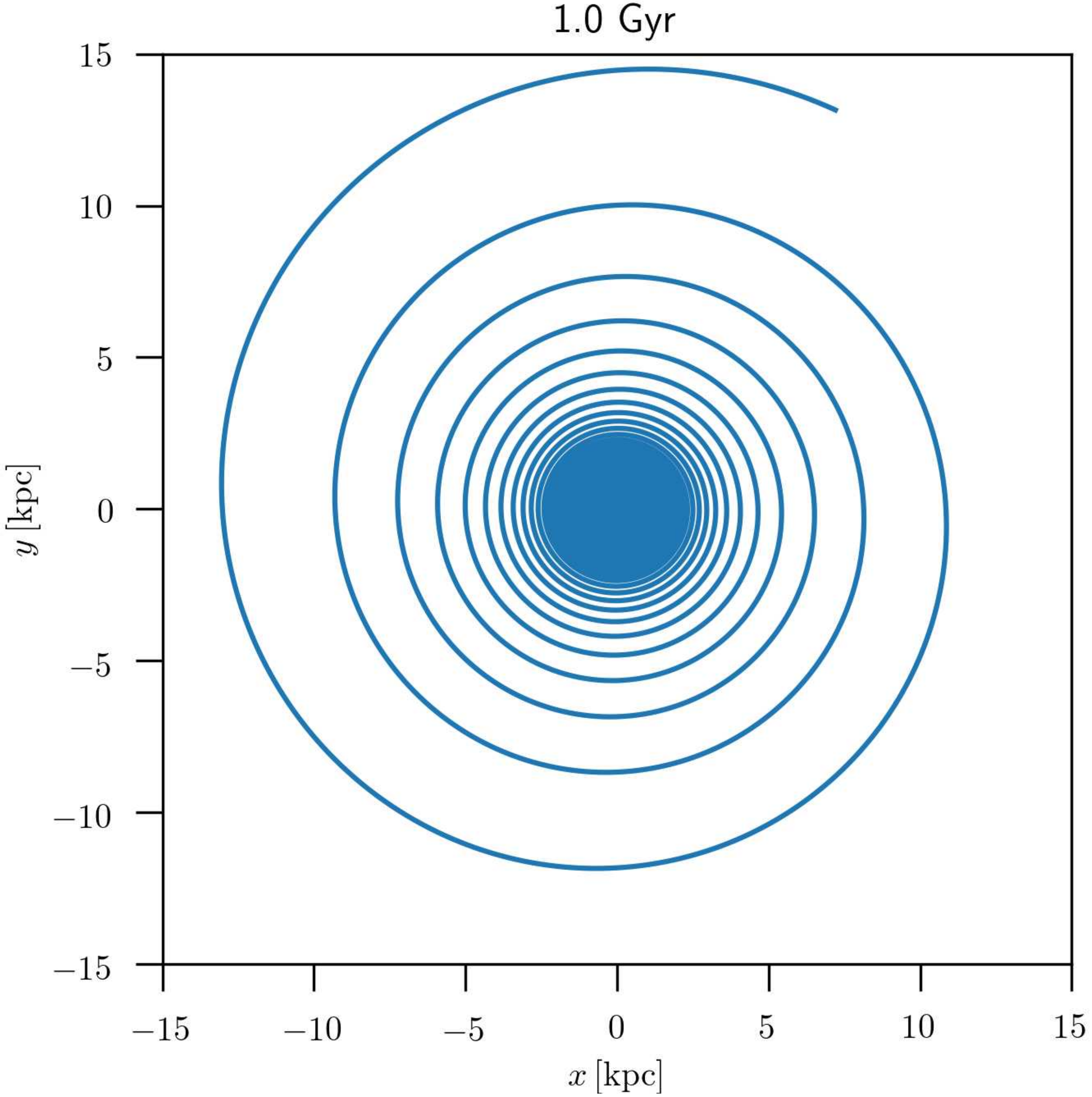
Winding problem : V=200 km/s



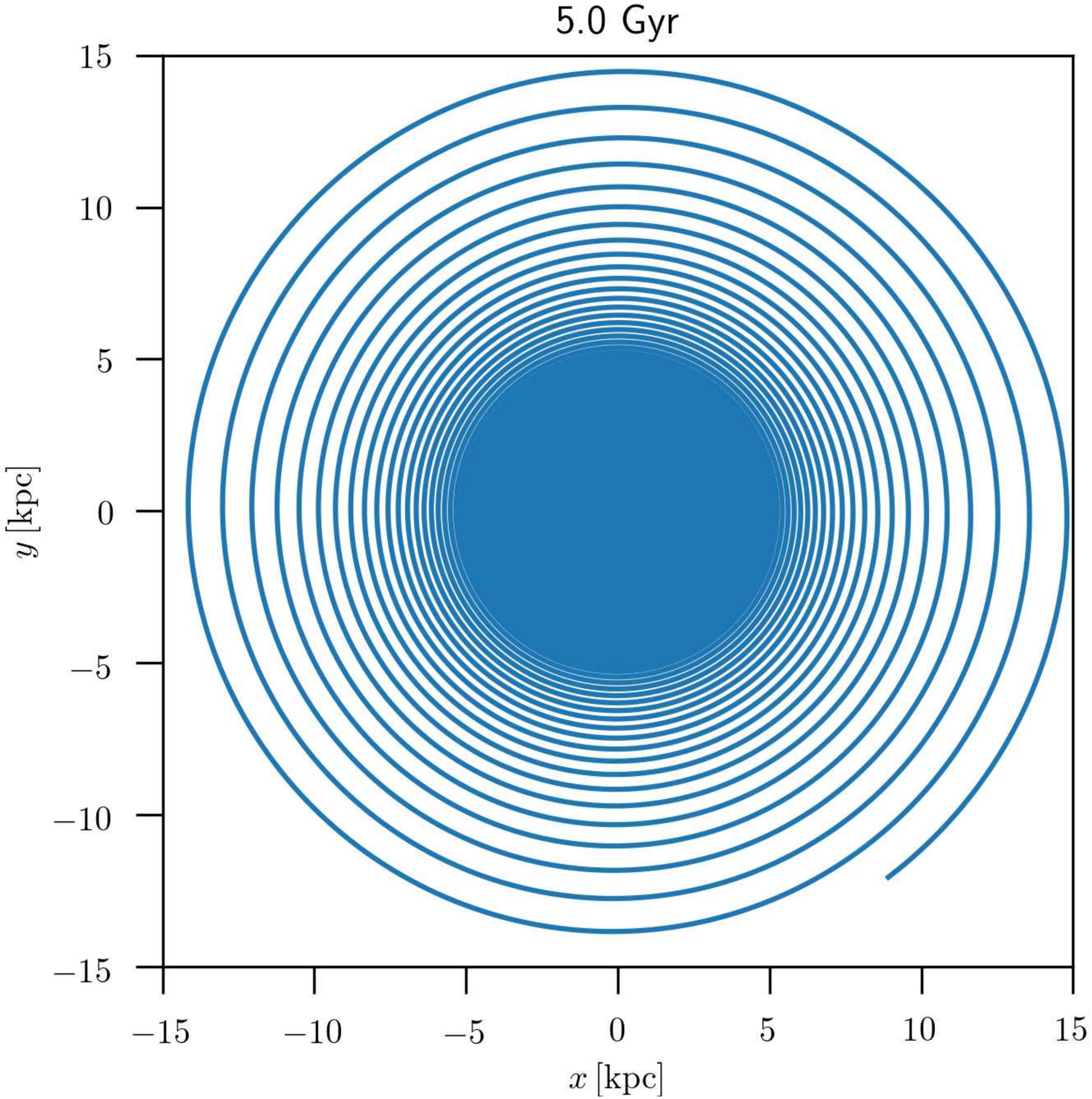
Winding problem : V=200 km/s



Winding problem : V=200 km/s



Winding problem : V=200 km/s



Possible solutions to the winding problem

1. A spiral arm is a transient phenomena, but the spiral pattern is statistically in a steady state.

   Example : a spiral arm traces young stars, until they die off
   → α could be much bigger

   Could explain flocculent galaxies, but not grand-design ones

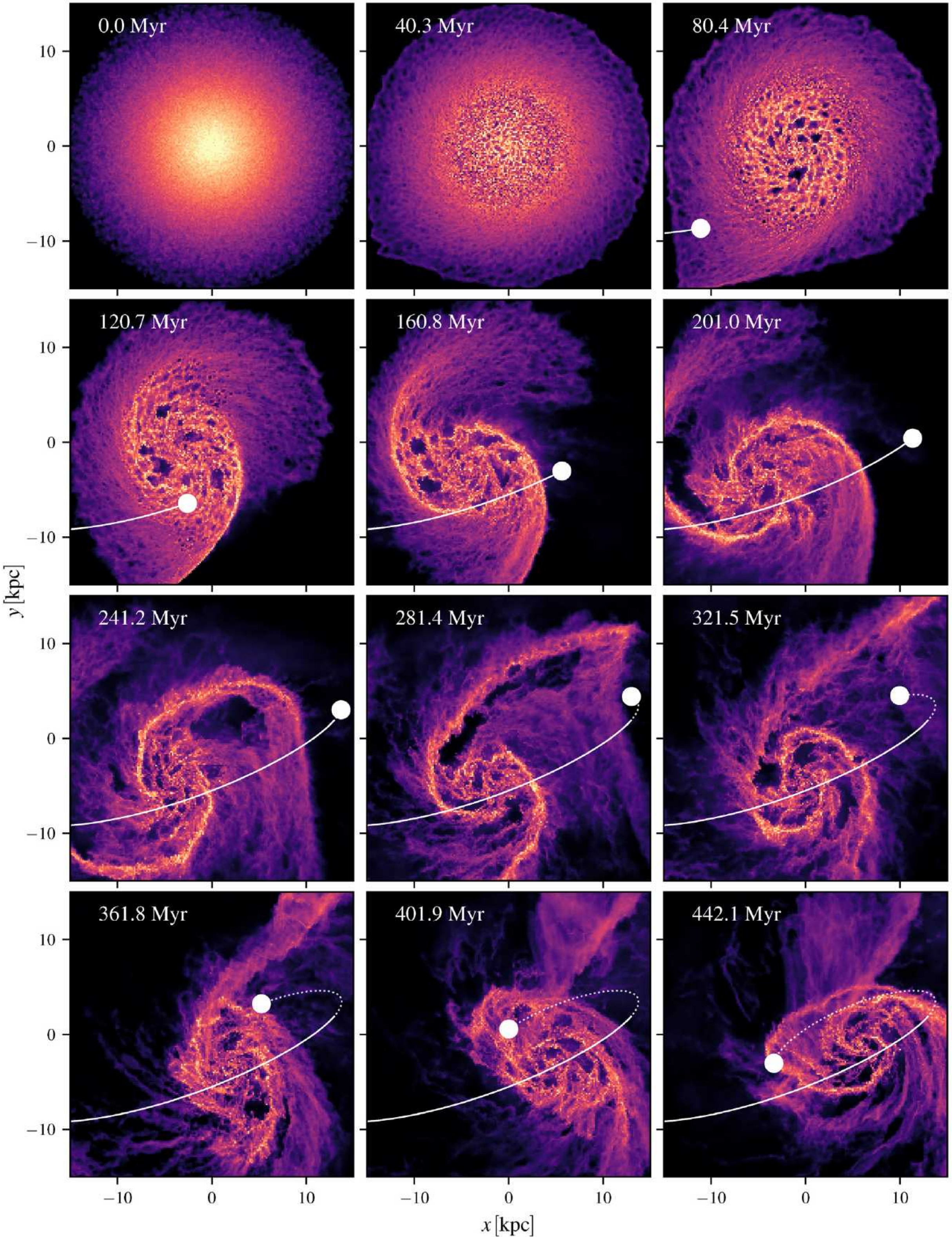
Possible solutions to the winding problem

1. A spiral arm is a transient phenomena, but the spiral pattern is statistically in a steady state.

   Example : a spiral arm traces young stars, until they die off
   → α could be much bigger

   Could explain flocculent galaxies, but not grand-design ones

2. The spiral pattern is a temporary phenomena, resulting for example from a recent event, like a merger or an interaction.


0.0 Myr
40.3 Myr
80.4 Myr
120.7 Myr
160.8 Myr
201.0 Myr
241.2 Myr
281.4 Myr
321.5 Myr
361.8 Myr
401.9 Myr
442.1 Myr
10
0
−10
y [kpc]
−10
0
10
x [kpc]

Tress 2020

Possible solutions to the winding problem

1. A spiral arm is <u>a transient phenomena</u>, but the spiral pattern is statistically in a steady state.

   <u>Example</u> : a spiral arm traces young stars, until they die off
   → α could be much bigger

   Could explain flocculent galaxies, but not grand-design ones

2. The spiral pattern is <u>a temporary phenomena</u>, resulting for example from a recent event, like a merger or an interaction.

3. The spiral structure is <u>a stationary density wave</u> that rotate rigidly in ρ and ϕ and thus, not subject to the winding problem (Lin-Shu 1964).
   The spiral pattern is a kind a mode, like a drum that vibrate.

Stability of collisionless systems

The stability of rotating disks :

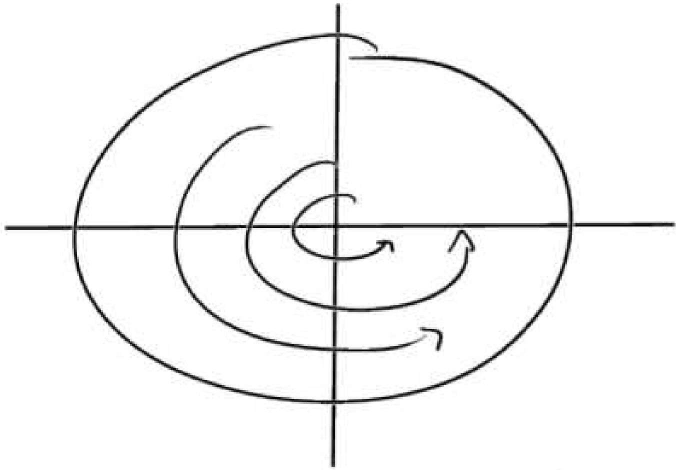
the dispersion relation

(in a nutshell)

The dispersion relation for a razor thin rotating fluid disk

Polar coordinates in the inertial rest frame

$\Sigma_d(R,\phi)\,,\quad \phi_d(R,\phi)\,,\quad p(R,\phi)\,,\quad \vec{v}(R,\phi) = v_R(R,\phi)\,\vec{e_R} + v_\phi(R,\phi)\,\vec{e_\phi}$



the disc can have a differential rotation

① Continuity equation $\quad \dfrac{\partial \Sigma_d}{\partial t} + \vec{\nabla}(\Sigma_d \vec{v}) = 0$

$$\boxed{\frac{\partial \Sigma_d}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\,\Sigma_d\,v_R) + \frac{1}{R}\frac{\partial}{\partial \phi}(\Sigma_d\,v_\phi) = 0}$$

② **Euler equation** $\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\Sigma_d} - \vec{\nabla}\phi$

$$\boxed{\begin{aligned} &\frac{\partial v_R}{\partial t} + v_R \frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} = -\frac{\partial \phi}{\partial R} - \frac{1}{\Sigma_d}\frac{\partial p}{\partial R} \\ &\frac{\partial v_\phi}{\partial t} + v_R \frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_R}{R} = -\frac{1}{R}\frac{\partial \phi}{\partial \phi} - \frac{1}{\Sigma_d}\frac{1}{R}\frac{\partial p}{\partial \phi} \end{aligned}}$$

③ **Poisson** $\boxed{\nabla^2 \phi = 4\pi G \Sigma \delta(z)}$

④ **Equation of state** (polytropic)

$\boxed{p = K \Sigma_d^{\gamma}}$ $\boxed{v_s^2 = \gamma K \Sigma_0^{\gamma-1}}$ (unperturbed sound speed)

The response of the system to a weak perturbation $-\varepsilon \vec{\nabla} \phi_e$

$$\Sigma_{d0} \rightarrow \Sigma_{d0} + \varepsilon \Sigma_{d1}$$

$$v_{R_0} = 0 \rightarrow \varepsilon v_{R1}$$

$$v_{\phi_0} \rightarrow v_{\phi_0} + \varepsilon v_{\phi_1}$$

$$h_0 \rightarrow h_0 + \varepsilon h_1$$

$$\phi_{d0} \rightarrow \phi_{d0} + \varepsilon \phi_{d1}$$

disk only with out the perturbation

$$\overset{\phi_{d0}}{\parallel}$$

$$\phi_0 \rightarrow \phi_{d0} + \underbrace{\varepsilon \phi_1}_{\varepsilon \phi_{d1} + \varepsilon \phi_e}$$

total potential, includes the perturbation

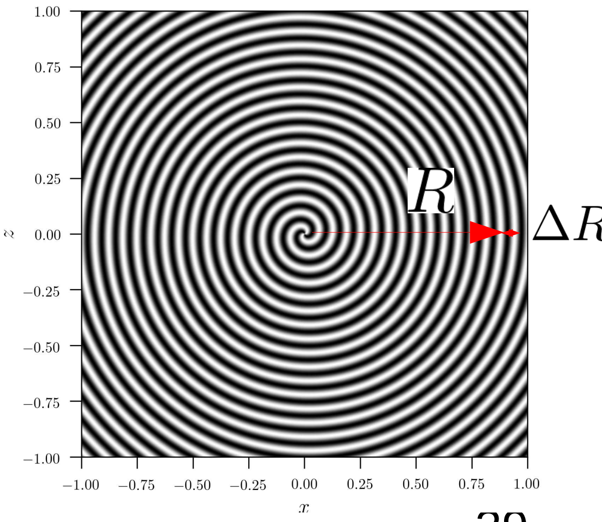
\+ solutions of the form

$$\Sigma_{d1}(R, \phi, t) \sim e^{i(m\phi + f(R,t) - \omega t)}$$

\+ WKB approximation to solve the Poisson equation

$$\phi_{d1} = - \frac{2\pi G}{|k|} e^{i(m\phi + f(R,t) - \omega t)}$$

$$k = \frac{\partial f}{\partial R}$$



We obtain the dispersion relation: (for $\Sigma_c = 0$)

$$\boxed{(\omega - m\Omega)^2 = \varkappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2}$$

$\varkappa$: epicycle radial frequency

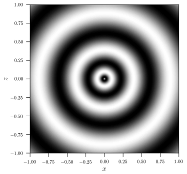
<u>Note</u>: if $\Omega(R) = \text{cte}$ $\quad \varkappa = 2\Omega$ and $m = 0$ we recover the dispersion relation for uniformly rotating sheet

$$\boxed{\omega^2 = 4\Omega^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2}$$

Interpretation

$$\boxed{(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2}$$

Axisymmetric perturbations $m = 0$



"Cold disk" $v_s = 0$

$$\boxed{\omega^2 = \kappa^2 - 2\pi G \Sigma_0 |k|}$$

$$k_{crit} := \frac{\kappa^2}{2\pi G \Sigma_0} \qquad \lambda_{crit} := \frac{2\pi}{k_{crit}} = \frac{4\pi^2 G \Sigma_0}{\kappa^2}$$

$$\boxed{\omega^2 = 2\pi G \Sigma_0 (k_{crit} - |k|)}$$

$|k| > k_{crit}$ $(\lambda < \lambda_{crit})$ $\omega^2 < 0$ **UNSTABLE**

$|k| < k_{crit}$ $(\lambda > \lambda_{crit})$ $\omega^2 > 0$ **STABLE**

The differential rotation stabilizes the system at large scale

$$\boxed{(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2}$$

"Non rotating fluid disk"

$\kappa = 0$

⚠ not realistic

$$\boxed{w^2 = -2\pi G \Sigma_0 |k| + v_s^2 k^2}$$

$$k_{crit} := \frac{2\pi G \Sigma_0}{v_s^2} \quad (\equiv k_{Jeans}) \qquad \lambda_{crit} = \frac{2\pi}{k_{crit}} = \frac{v_s^2}{G\Sigma_0}$$

$$\boxed{w^2 = v_s^2 |k| (|k| - k_{crit})}$$

$|k| > k_{crit}$ $(\lambda < \lambda_{crit})$ $w^2 > 0$ **STABLE**

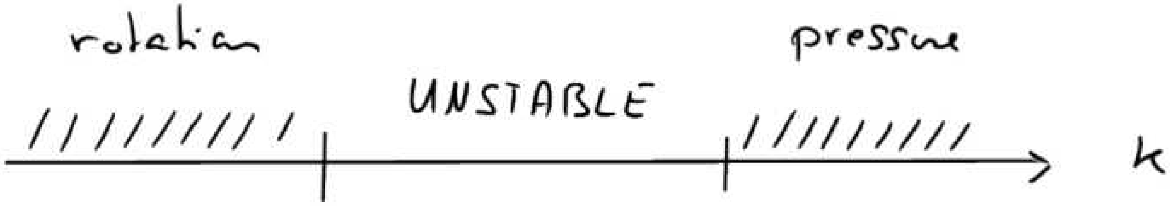
$|k| < k_{crit}$ $(\lambda > \lambda_{crit})$ $w^2 < 0$ **UNSTABLE**

The pressure stabilizes the system at small scale

$$\boxed{(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2}$$

Rotating fluid disk

$$\boxed{\omega^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2}$$



$$k_{c,1} = \frac{\kappa^2}{2\pi G \Sigma_0} \qquad k_{c,2} = \frac{2\pi G \Sigma_0}{v_s^2}$$

Complete stability

$$k_1 \geqslant k_2$$

$$\boxed{Q := \frac{\kappa v_s}{\pi G \Sigma_0} > 1}$$

Stability of stellar disks

(using a Schwarzschild DF)

$$\boxed{Q := \frac{\varkappa v_s}{\pi G \Sigma_0} > 1}$$

$$\boxed{Q := \frac{\varkappa \sigma_R}{3.36\, G \Sigma_0} > 1}$$

Safronov - Toomre

criterion

(1964)

Non axisymmetric perturbations

$$\boxed{(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2}$$

$$\omega = \sqrt{\kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2} + m\Omega$$

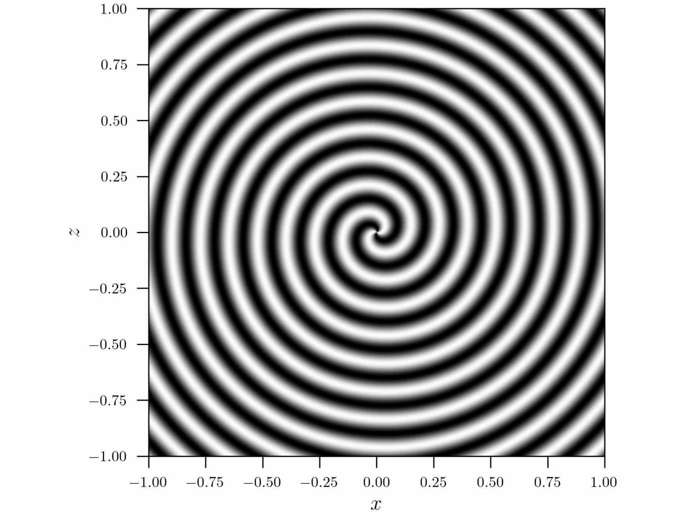
- the stability is determined for $m = 0$ (value of $\sqrt{}$)
- $m\Omega \in \mathbb{R}e$ → add an oscillatory term $e^{-i m\Omega}$
  with a frequency that correspond to
  the passage of spiral arm

Non axisymmetric perturbations

$$\boxed{(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2}$$

$$\omega = \sqrt{\kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2} + m\Omega$$

- the stability is determined for $m = 0$ (value of $\sqrt{\ }$)
- $m\Omega \in \mathrm{Re}$ → add an oscillatory term $e^{-im\Omega}$ with a frequency that correspond to the passage of spiral arm



Stability of collisionless systems

The stability of rotating disks :

the dispersion relation (details)

The dispersion relation for a razor thin fluid disk

polar coordinates

$$\Sigma_d(R,\phi)\,,\quad \phi_d(R,\phi)\,,\quad p(R,\phi)\,,\quad \vec{v}(R,\phi) = v_R(R,\phi)\,\vec{e}_R + v_\phi(R,\phi)\,\vec{e}_\phi$$

① Continuity equation

$$\frac{\partial \Sigma_d}{\partial t} + \vec{\nabla}(\Sigma_d \vec{v}) = 0$$

$$\boxed{\frac{\partial \Sigma_d}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\,\Sigma_d v_R) + \frac{1}{R}\frac{\partial}{\partial \phi}(\Sigma_d v_\phi)}$$

② Euler equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\Sigma_d} - \vec{\nabla}\phi$$

$$\boxed{\begin{aligned}
&\frac{\partial v_R}{\partial t} + v_R\frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} = -\frac{\partial \phi}{\partial R} - \frac{1}{\Sigma_d}\frac{\partial p}{\partial R}\\
&\frac{\partial v_\phi}{\partial t} + v_R\frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_R}{R} = -\frac{1}{R}\frac{\partial \phi}{\partial \phi} - \frac{1}{\Sigma_d}\frac{1}{R}\frac{\partial p}{\partial \phi}
\end{aligned}}$$

③ Poisson $\nabla^2 \phi = 4\pi G \Sigma \delta(z)$

④ Equation of state (polytropic)

$$p = K \Sigma_d^{\gamma} \qquad v_s^2 = \gamma K \Sigma_0^{\gamma-1} \quad \text{(unperturbed sound speed)}$$

$$h = \frac{\gamma}{\gamma-1} K \Sigma_d^{\gamma-1} \qquad \frac{\partial h}{\partial R} = \gamma K \Sigma_d^{\gamma-2} \frac{\partial \Sigma}{\partial R}$$

(Specific enthalpy)

$$\frac{\partial h}{\partial \phi} = \gamma k \Sigma_d^{\gamma-2} \frac{\partial \Sigma}{\partial \phi}$$

The Euler Equation becomes

$$\begin{cases} \dfrac{\partial V_R}{\partial t} + V_R \dfrac{\partial V_R}{\partial R} + \dfrac{V_\phi}{R} \dfrac{\partial V_R}{\partial \phi} - \dfrac{V_\phi^2}{R} = -\dfrac{\partial}{\partial R}\left(\phi + h \right) \\[2ex] \dfrac{\partial V_\phi}{\partial t} + V_R \dfrac{\partial V_\phi}{\partial R} + \dfrac{V_\phi}{R} \dfrac{\partial V_\phi}{\partial \phi} + \dfrac{V_\phi V_R}{R} = -\dfrac{1}{R} \dfrac{\partial}{\partial \phi}\left(\phi + h \right) \end{cases}$$

Isolated system at equilibrium

Σ_{d0} , ϕ_0 , p_0 , $\vec{v}_0$ + axisymmetric

+ $v_r = 0$ $\left(\frac{\partial}{\partial\phi} = 0\right)$

① Continuity equation $\quad 0 = 0$

② Euler equation $\quad \phi: \quad 0 = 0$

$$R: \quad \boxed{\underbrace{\frac{v_{\phi_0}^2}{R}}_{\text{centripetal acceleration}} = \frac{\partial}{\partial R}\left(\phi_0 + h_0\right) = \underbrace{\frac{\partial \phi_0}{\partial R}}_{\text{gravity force}} + \overbrace{\underbrace{v_s^2 \frac{d}{dR}\ln\Sigma_0}_{\text{pressure force}}}^{\circledast}}$$

Notes

① $\circledast$ $v_s^2 \frac{d}{dR}\ln\Sigma_0 = \left.\frac{\partial P}{\partial \Sigma}\right|_{\Sigma_0} \frac{1}{\Sigma_0}\frac{\partial \Sigma_0}{\partial R} = \frac{1}{\Sigma_0}\frac{\partial P}{\partial R} = \frac{1}{\Sigma_0}\vec{\nabla} P$

② $\left.\begin{array}{l} V_s \simeq 10 \text{ km/s} \\ V_\phi \simeq 100 \text{ km/s} \end{array}\right\} \Rightarrow$ we neglect the pressure

③ $V_{\phi_0} \simeq \sqrt{R \frac{\partial \phi_0}{\partial R}} = R\,\Omega(R)$

The response of the system to a weak perturbation $-\varepsilon \vec{\nabla} \phi_e$

$$\left.\begin{aligned}
\Sigma_{d0} &\rightarrow \Sigma_{d0} + \varepsilon \Sigma_{d1} \\
v_{R_0} = 0 &\rightarrow \varepsilon v_{R1} \\
v_{\phi_0} &\rightarrow v_{\phi_0} + \varepsilon v_{\phi_1} \\
h_0 &\rightarrow h_0 + \varepsilon h_1 \\
\phi_{d0} &\rightarrow \phi_{d0} + \varepsilon \phi_{d1}
\end{aligned}\right\}$$

disk only without the perturbation

$$\shortparallel$$

$$\left.\phi_0 \rightarrow \phi_{d0} + \underbrace{\varepsilon \phi_1}_{\varepsilon \phi_{d1} + \varepsilon \phi_e}\right\}$$

total potential, includes the perturbation

Linearized equations for a razor thin fluid disk

① Continuity equation

$$\boxed{\frac{\partial}{\partial t}\Sigma_{d1} + \Omega\frac{\partial \Sigma_{d1}}{\partial \phi} + \frac{1}{R}\frac{\partial}{\partial R}\left(R v_{R1}\Sigma_0\right) + \frac{\Sigma_0}{R}\frac{\partial v_{\phi 1}}{\partial \phi} = 0}$$

② Euler equation

$$\boxed{\begin{aligned} &\frac{\partial v_{R1}}{\partial t} + \Omega\frac{\partial v_{R1}}{\partial \phi} - 2\Omega v_{\phi 1} = -\frac{\partial}{\partial R}\left(\Phi_1 + h_1\right) \\ &\frac{\partial v_{\phi 1}}{\partial t} + \underbrace{\left[\frac{\partial}{\partial R}(\Omega R) + \Omega\right]}_{-2B(R)} v_{R1} + \Omega\frac{\partial v_{\phi 1}}{\partial \phi} = -\frac{1}{R}\frac{\partial}{\partial \phi}\left(\Phi_1 + h_1\right)\end{aligned}}$$

Oort constant

$\kappa^2 = -4B\Omega$

Solutions

assume waves of the form (can be a spiral)

5 radial functions

$$\left.\begin{aligned}
\underline{\Sigma_{d1}} &= \mathrm{Re}\left[\Sigma_{da}(R)\, e^{i(m\phi-\omega t)}\right] \\
\underline{v_{R1}} &= \mathrm{Re}\left[v_{Ra}(R)\, e^{i(m\phi-\omega t)}\right] \\
\underline{v_{\phi 1}} &= \mathrm{Re}\left[v_{\phi a}(R)\, e^{i(m\phi-\omega t)}\right] \\
\underline{h_1} &= \mathrm{Re}\left[h_a(R)\, e^{i(m\phi-\omega t)}\right]
\end{aligned}\right\}$$

without perturbation

$$\left.\begin{aligned}
\Sigma_1 &= \mathrm{Re}\left[\Sigma_a(R)\, e^{i(m\phi-\omega t}\right] \\
\underline{\phi_1} &= \mathrm{Re}\left[\phi_a(R)\, e^{i(m\phi-\omega t}\right]
\end{aligned}\right\}$$

total: disk + perturbation

$$\Sigma_a = \Sigma_{da} + \Sigma_e$$

$$\phi_a = \phi_{da} + \phi_e$$

① The continuity equation gives

$$\boxed{-i(\omega - m\Omega)\underline{\Sigma_{da}} + \frac{1}{R}\frac{d}{dR}\left(R\,\underline{v_{Ra}}\,\Sigma_0\right) + \frac{im\Sigma_0}{R}\underline{v_{\phi a}} = 0}$$

② The Euler equation gives

$$\boxed{\begin{cases} \underline{v_{R_a}}(R) = \frac{i}{\Delta}\left[(\omega - m\Omega)\frac{d}{dR}(\underline{\phi_a} + \underline{h_a}) - \frac{2m\Omega}{R}(\underline{\phi_a} + \underline{h_a}) \right] \\ \underline{v_{\phi_a}}(R) = -\frac{i}{\Delta}\left[2B\frac{d}{dR}(\underline{\phi_a} + \underline{h_a}) + \frac{m(\omega - m\Omega)}{R}(\underline{\phi_a} + \underline{h_a}) \right] \end{cases}}$$

with $\underline{\Delta = \kappa^2 - (\omega - m\Omega)^2}$

Notes ① v_{R_a}, v_{ϕ_a}, κ, Δ, Ω are functions of R only

② if $\Delta = 0 \Rightarrow$ divergeance

$\Omega_p = \frac{\omega}{m}$ pattern speed

$$\kappa^2 = (\omega - m\Omega)^2$$

$$= m^2\left(\frac{\omega}{m} - \Omega\right)^2 = m^2(\Omega_p - \Omega)^2$$

$$\underline{\Omega_p = \Omega \pm \frac{\kappa}{m}}$$

!!! problems at the Lindblad resonance

④ "Equation of state" $h = \frac{\gamma}{\gamma-1} K \Sigma_d^{\gamma-1}$ $v_s^2 = \gamma K \Sigma_0^{\gamma-1}$

$$\boxed{\underline{h_a} = \gamma K \Sigma_0^{\gamma-2} \underline{\Sigma_{da}} = v_s^2 \frac{\underline{\Sigma_{da}}}{\Sigma_0}}$$

=> we have 4 equations for 5 unknowns $\Sigma_{da}, h_a, \phi_a, v_{Ra}, v_{\phi a}$

⑤ Poisson Equation use WKB approximation

$$\Sigma_1 = R_e\left[\Sigma_a(R)\, e^{i(m\phi-\omega t)}\right]$$

$$= R_e\left[e^{-i\omega t}\, \Sigma_a(R)\, e^{im\phi}\right]$$

$$= R_e\left[H(R,t)\, e^{iS(R,t)}\, e^{im\phi}\right]$$

$$\boxed{\begin{array}{c} k = \frac{\partial S}{\partial R} \\ |kR| \gg 1 \end{array}}$$

$$\phi_1 = -\frac{2\pi G}{k} \underbrace{H(R,t)}_{\text{slow variation}} \underbrace{e^{iS(R,t)}}_{\text{fast variation}} \underbrace{e^{im\phi}}_{\text{azimuthal symmetry}} \Rightarrow \boxed{\Sigma_{da} \sim \phi_a}$$

In addition

④ "Eq. of state" $\Rightarrow \quad h_a \sim \Sigma_{da} \sim \phi_1 \sim e^{if(R,t)}$

Thus a) $\frac{d}{dR}(\phi_a + h_a) \sim (\phi_a + h_a)\, i \frac{d}{dR} f = (\phi_a + h_a)\, ik$

b) $\frac{1}{R}(\phi_a + h_a) \cong \frac{1}{R}(\phi_a + h_a)\, ik \frac{1}{ik} = \frac{-i}{Rk} \frac{d}{dR}(\phi_a + h_a)$

as $Rk \gg 1$ $\quad \underline{\frac{1}{R}(\phi_a + h_a) \ll \frac{d}{dR}(\phi_a + h_a)}$

① The continuity equation becomes $\quad \frac{d}{dR}(R \Sigma_0 v_{Ra}) = ik R \Sigma_0 v_{Ra}$

$$\boxed{-(\omega - m\Omega)\,\underline{\Sigma_{da}} + k \Sigma_0 \underline{v_{Ra}} = 0}$$

② The Euler equation becomes $\quad + \underset{(h_a \sim \Sigma_{da})}{\text{EoS}} + \underset{(\Sigma_a \sim \phi_a)}{\text{poisson}}$

$$\boxed{\begin{aligned} \underline{v_{Ra}} &= - \frac{\omega - m\Omega}{\Delta} k (\underline{\phi_a} + \underline{h_a}) \\ \underline{v_{\phi a}} &= - \frac{2iB}{\Delta} k (\underline{\phi_a} + \underline{h_a}) \end{aligned}}$$

We can solve to get

$$\boxed{\Sigma_{da} = \left(\frac{2\pi G \Sigma_0 |k|}{\kappa^2 - (\omega - m\Omega)^2 + v_s^2 k^2} \right) \Sigma_a}$$

Which gives the dispersion relation if $\Sigma_e = 0$ $(\Sigma_{da} = \Sigma_a)$

$$\boxed{(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2}$$

Note: if $\Omega(R) = \text{cte}$ $\kappa = 2\Omega$ and we recover the dispersion relation for uniformly rotating sheet

$$\boxed{\omega^2 = 4\Omega^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2}$$

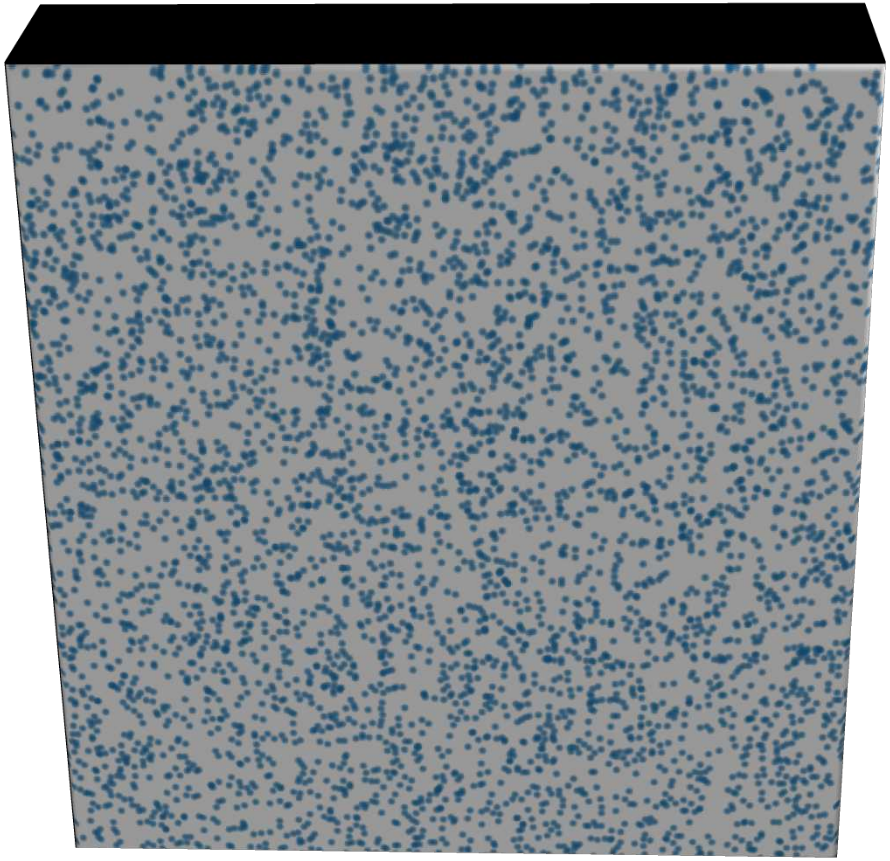
Stability of collisionless systems

The origin of spiral structures :

another view

The Jeans instability

(Jeans 1902)



- Infinite slab of infinite thickness, homogeneous Σ
- Particles with random velocities (constants vel. dispersion σ)
- Gravitational interactions only (collisonless system).

The Jeans instability

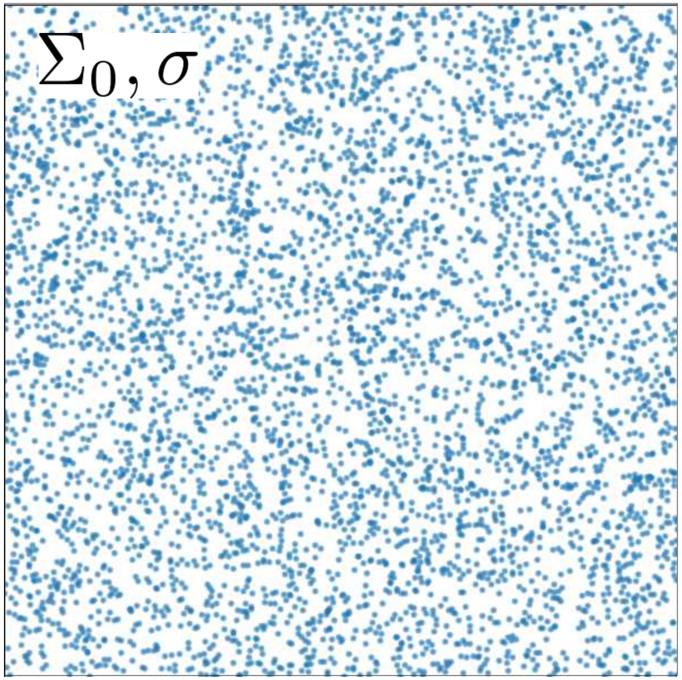
(Jeans 1902)

- Infinite razor thin medium

$$\lambda_{\rm J} = 2\,r_{\rm J} = \frac{\sigma^2}{G\Sigma_0}$$

$$\lambda > \lambda_J \qquad\qquad \lambda < \lambda_J$$

unstable stable



The Jeans instability

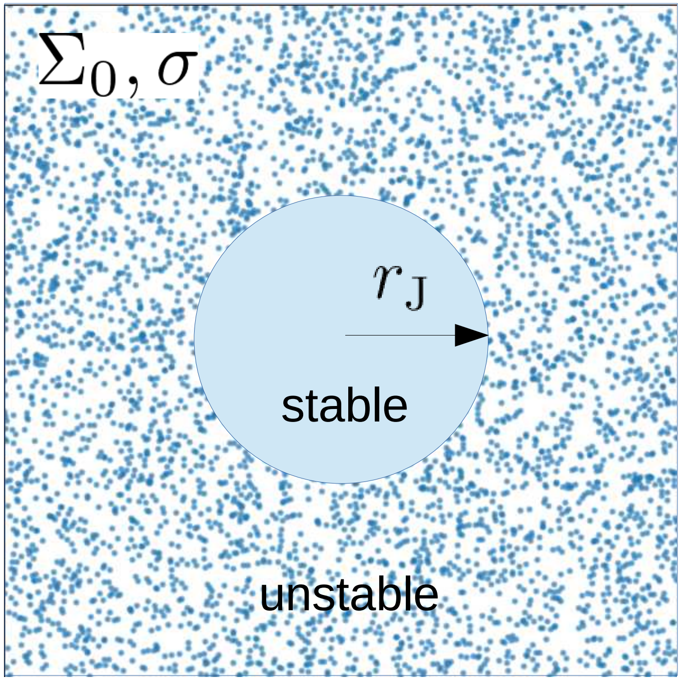
(Jeans 1902)

- Infinite razor thin medium

$$\lambda_\mathrm{J} = 2\,r_\mathrm{J} = \frac{\sigma^2}{G\Sigma_0}$$

$r > r_J$ $r < r_J$

<u>unstable</u> <u>stable</u>



The link with the virial equilibrium

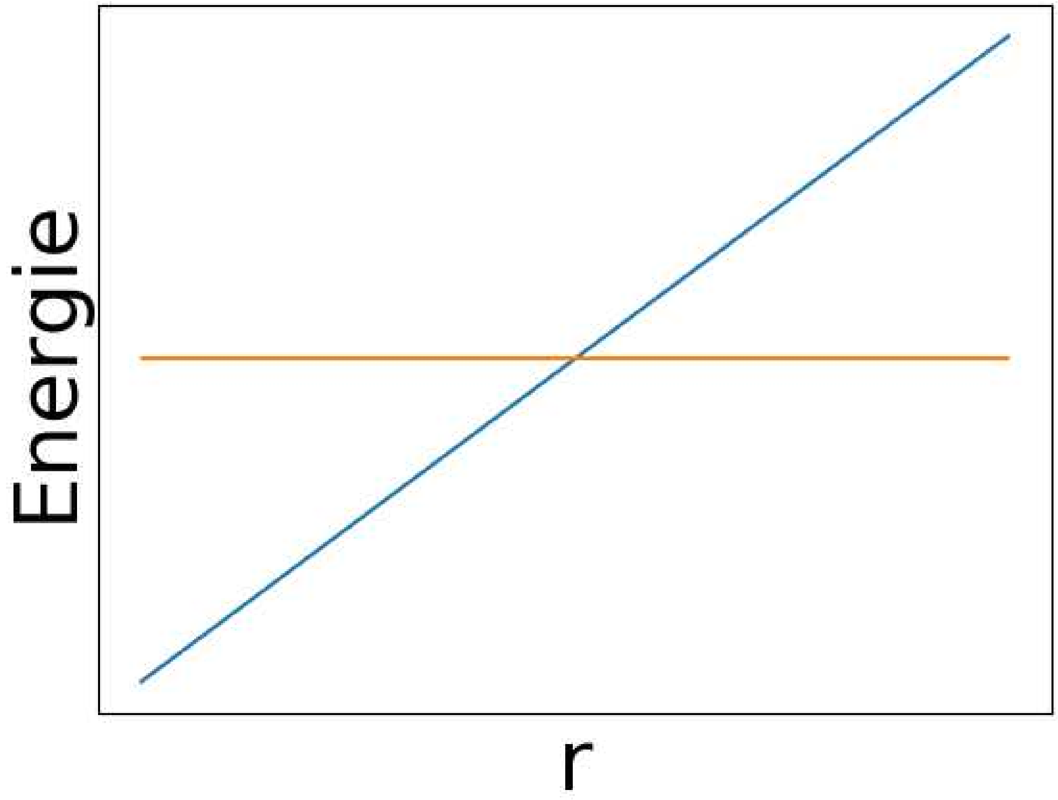
$$\boxed{\sigma^2 = \frac{2}{\pi}\frac{GM_\mathrm{J}}{r_\mathrm{J}}}$$

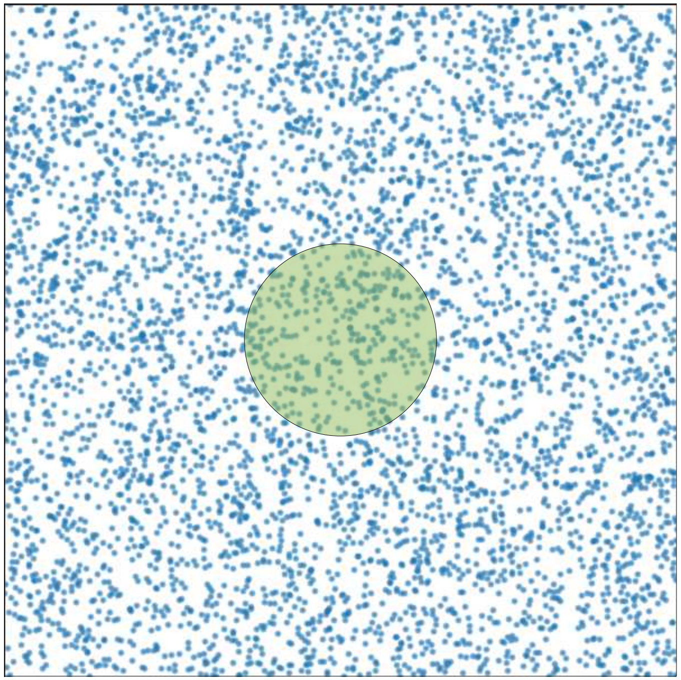
with

$$M_\mathrm{J} = \pi r_\mathrm{J}^2\Sigma_0$$

The Jeans instability

(Jeans 1902)

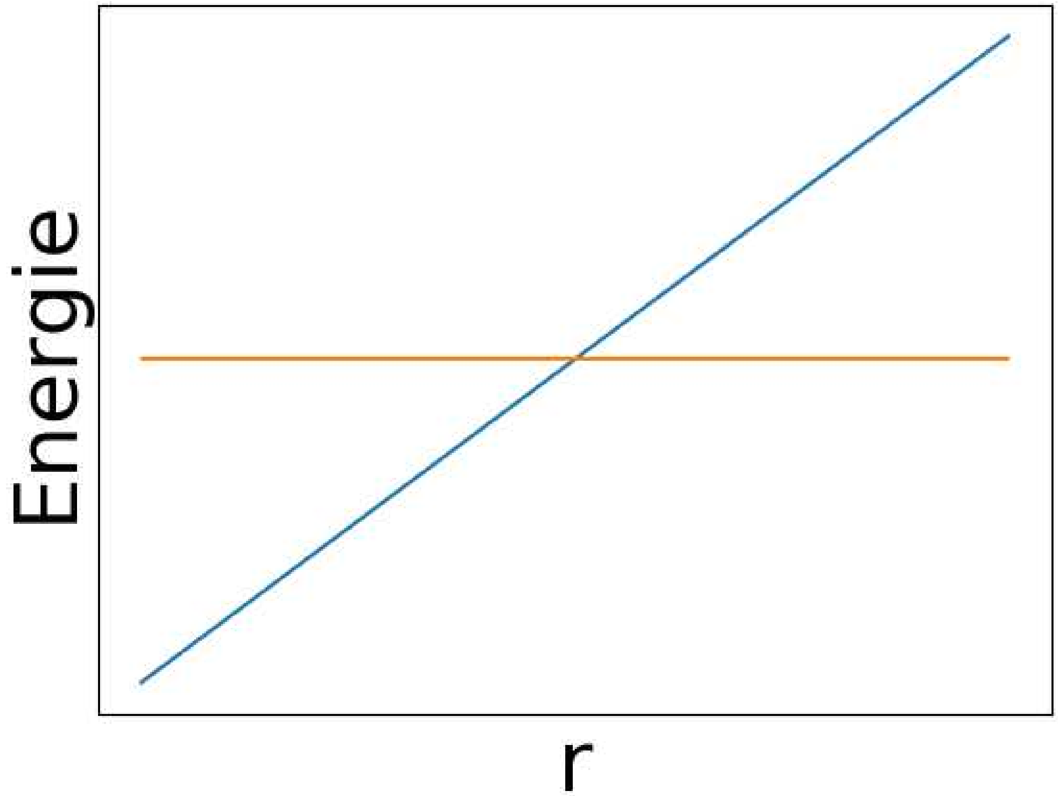


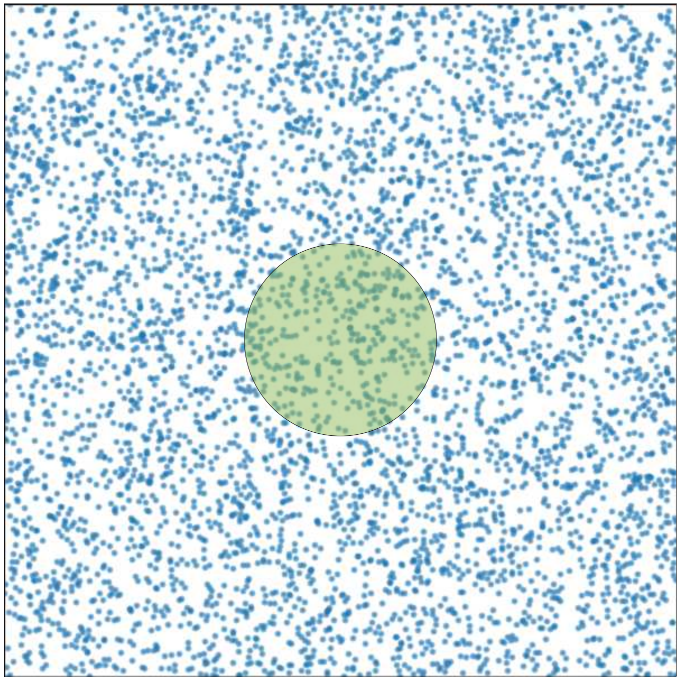


$$K \sim \sigma^2$$

The Jeans instability

(Jeans 1902)

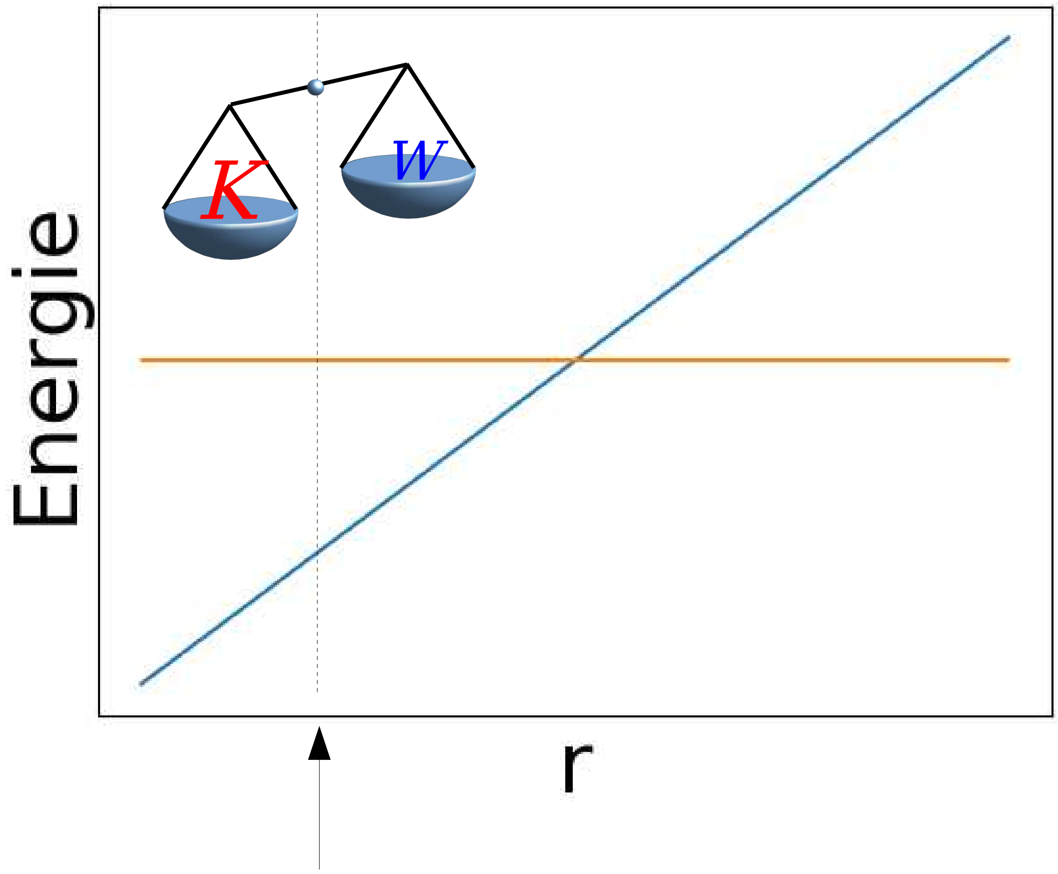


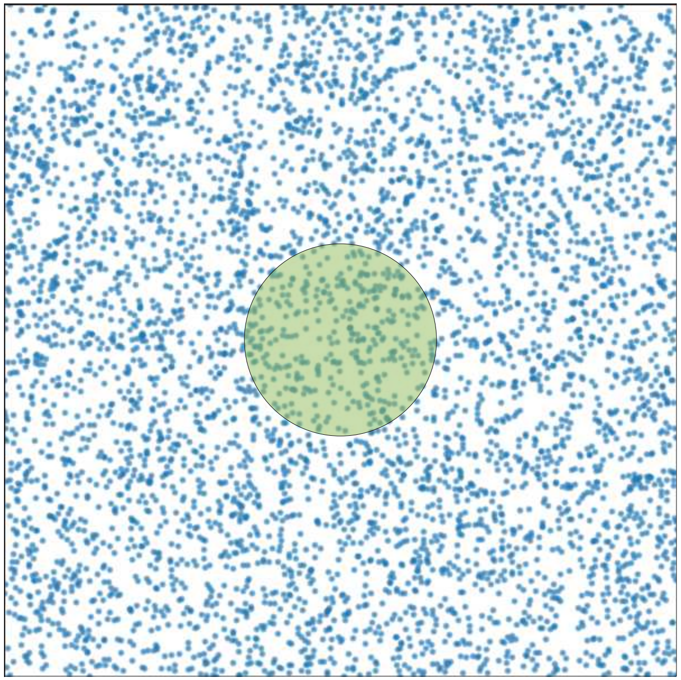


$$K \sim \sigma^2 \qquad W \sim \frac{G\,M(r)}{r} = G\Sigma\pi r$$

The Jeans instability

(Jeans 1902)

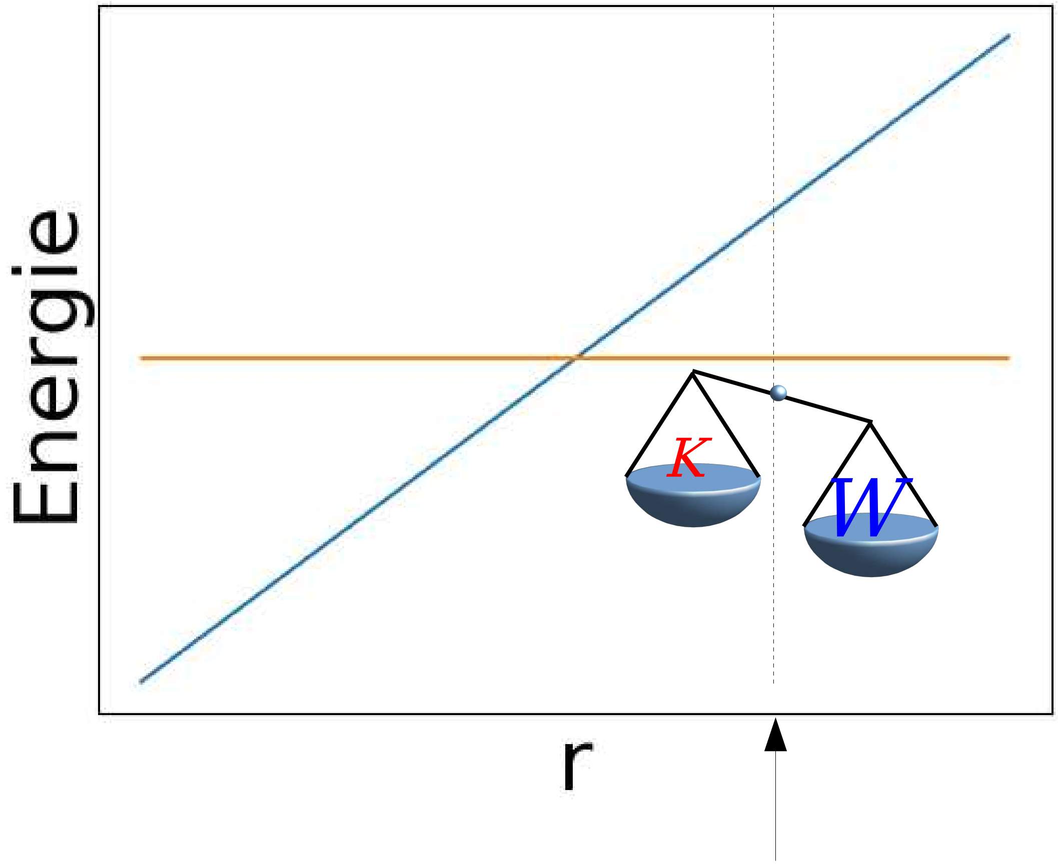


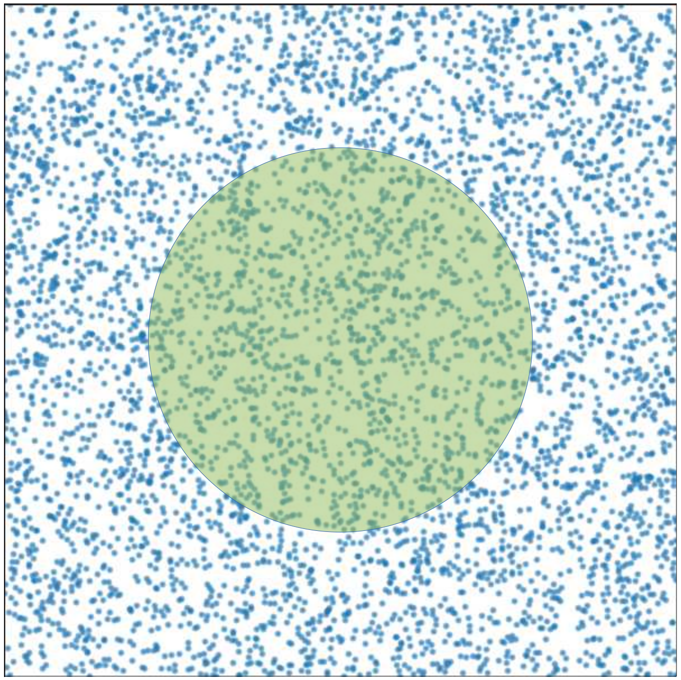


$$K \sim \sigma^2 \qquad W \sim \frac{G\,M(r)}{r} = G\Sigma\pi r$$

The Jeans instability

(Jeans 1902)





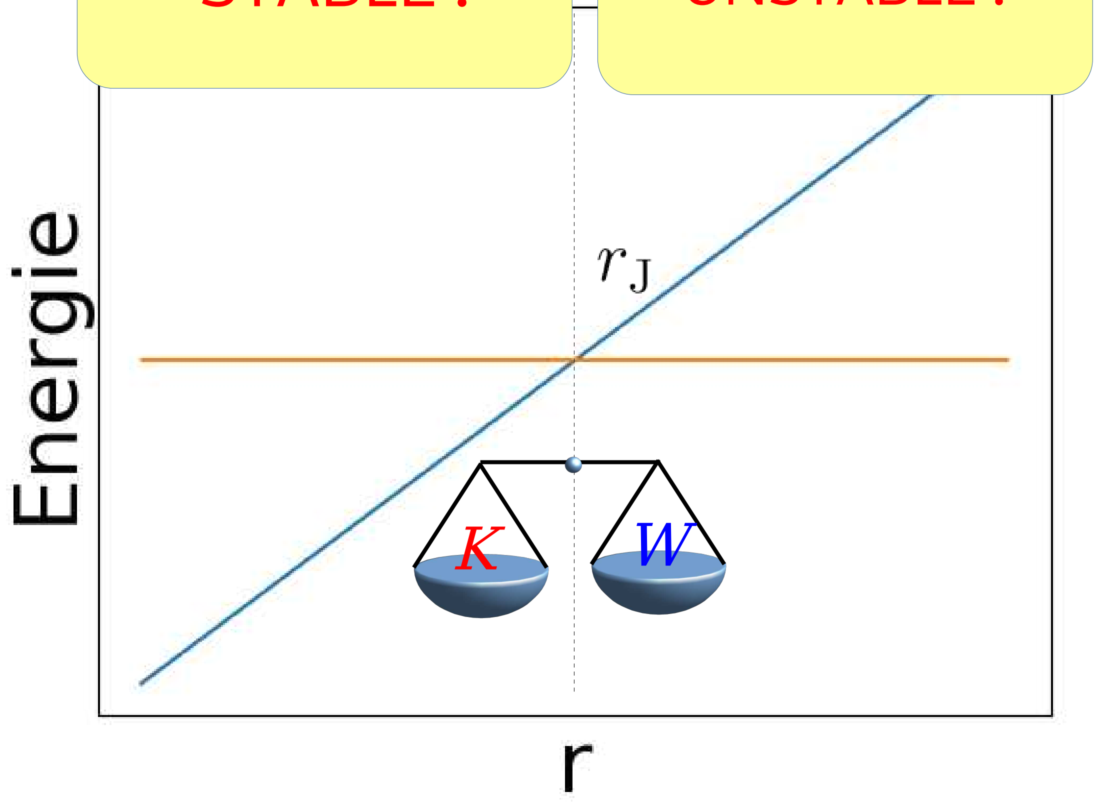
$$K \sim \sigma^2 \qquad W \sim \frac{G\,M(r)}{r} = G\Sigma\pi r$$

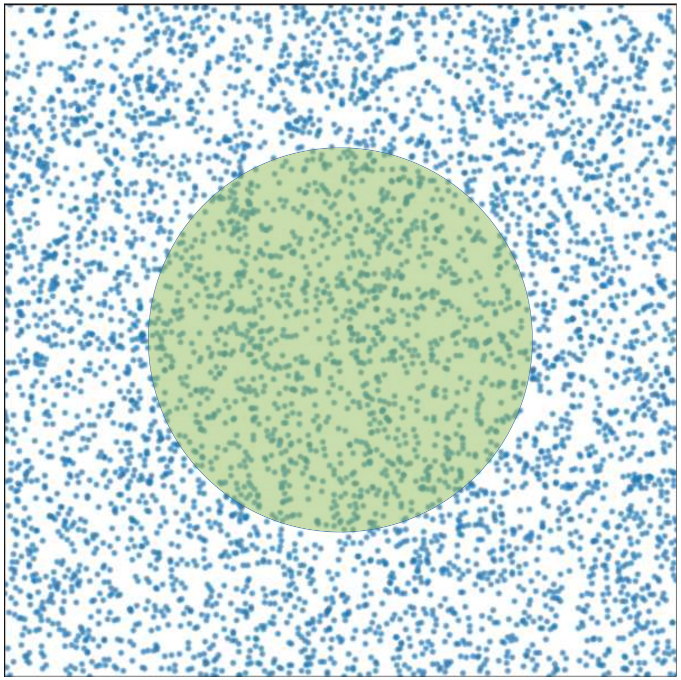
The Jeans instability

(Jeans 1902)

STABLE !

UNSTABLE !

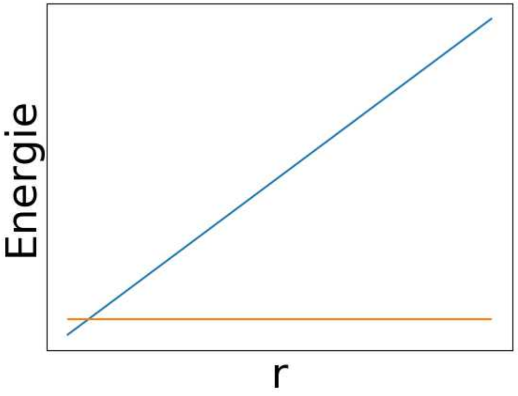




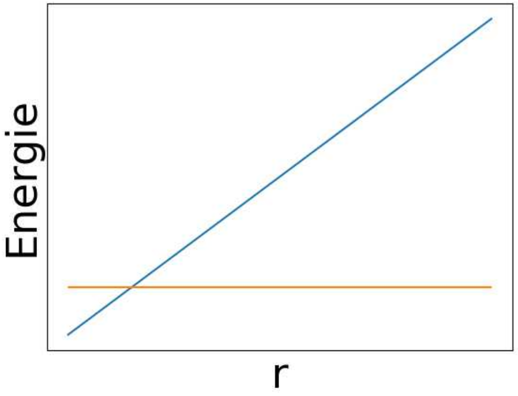
$$K \sim \sigma^2 \qquad W \sim \frac{G\,M(r)}{r} = G\Sigma\pi r$$

The Jeans instability in an infinite razor-thin sheet

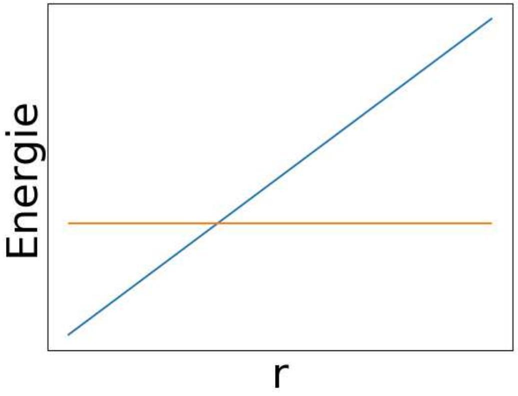
$\sigma = 0.1, r_\mathrm{J} = 0.005$

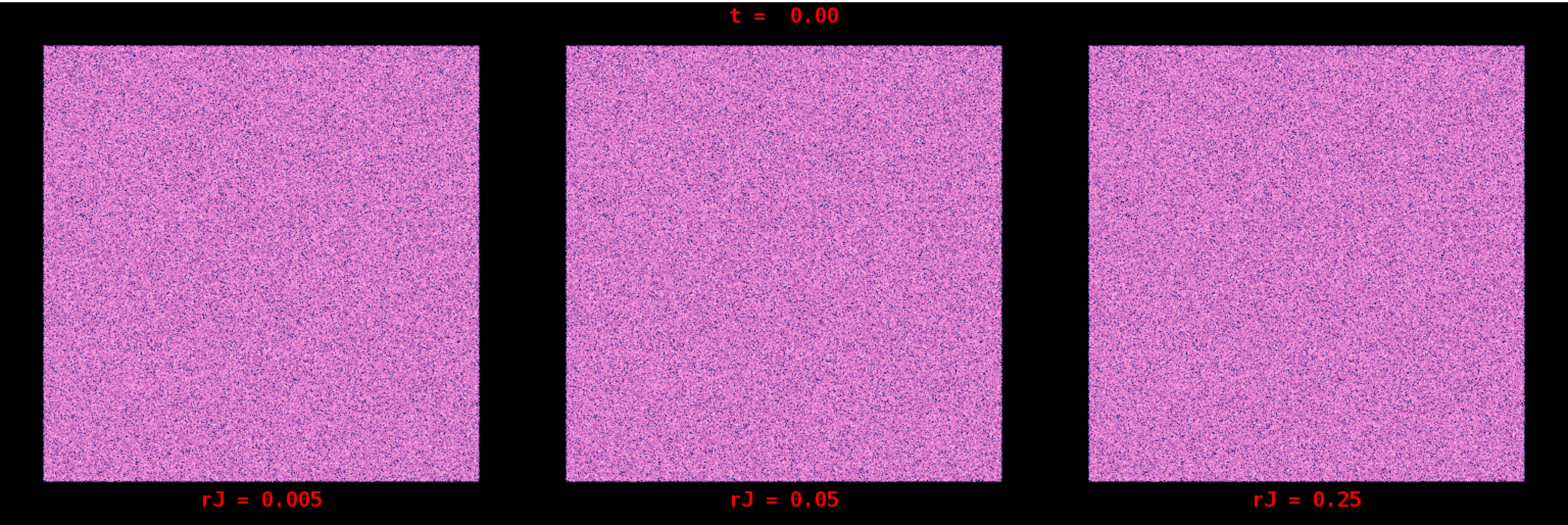


$\sigma = 0.3, r_\mathrm{J} = 0.05$



$\sigma = 0.7, r_\mathrm{J} = 0.25$

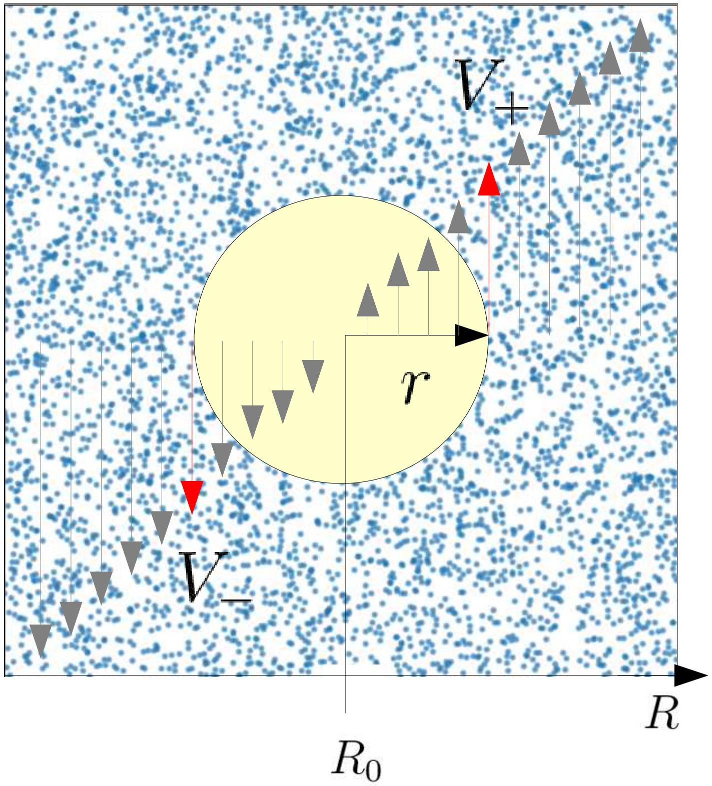




Can we stabilize a razor-thin sheet against gravitational instabilities using non-random motions ?

$$2T = K$$

Stabilizing using shearing motions



$V(R)$: a vertical velocity field (aligned with the y axis)

At the edges of a disk

$$V_+ = V(R_0) + \left(\frac{\mathrm{dV}}{\mathrm{dR}} \bigg|_{R_0} \right) r$$

$$V_- = V(R_0) - \left(\frac{\mathrm{dV}}{\mathrm{dR}} \bigg|_{R_0} \right) r$$

- Kinetic energy

$$2\, E_{\mathrm{kin}} \cong \Delta V^2 = (V_+ - V_-)^2 = 4 \left(\frac{\mathrm{dV}}{\mathrm{dR}} \bigg|_{R_0} \right)^2 r^2$$

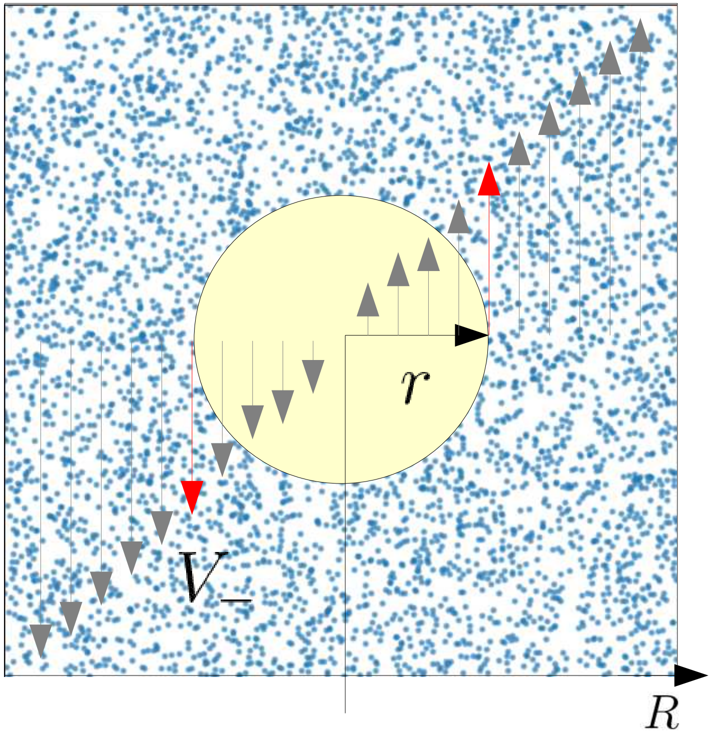
Stabilizing using shearing motions

- Kinetic energy

$$2\,E_{\mathrm{kin}} = \frac{1}{\pi\,r^2}\int_S V(R)^2 ds$$

$$V(R)^2 \cong V^2(R_0) + \left(\frac{\mathrm{dV}}{\mathrm{dR}}\bigg|_{R_0}\right)^2 (R-R_0)^2 + 2\,V(R_0)\,\frac{\mathrm{dV}}{\mathrm{dR}}\bigg|_{R_0}(R-R_0)$$

$$2\,E_{\mathrm{kin}} = V^2(R_0) + \frac{1}{2}\left(\frac{\mathrm{dV}}{\mathrm{dR}}\bigg|_{R_0}\right)^2 r^2$$



Stabilizing using shearing motions

- Kinetic energy

$$2\,E_{\mathrm{kin}} = \frac{1}{\pi\, r^2}\int_S V(R)^2 ds$$

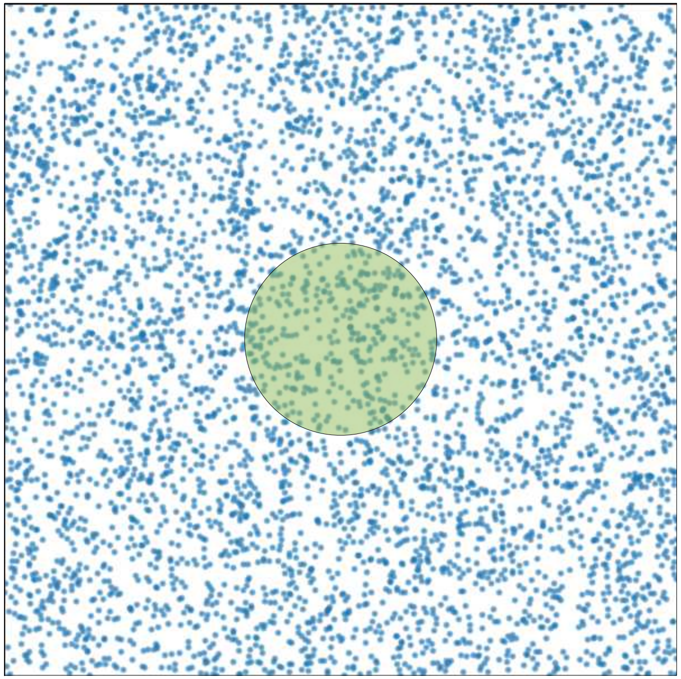
$$V(R)^2 \cong V^2(R_0) + \left(\frac{\mathrm{dV}}{\mathrm{dR}}\bigg|_{R_0}\right)^2 (R-R_0)^2 + 2\,V(R_0)\,\frac{\mathrm{dV}}{\mathrm{dR}}\bigg|_{R_0}(R-R_0)$$

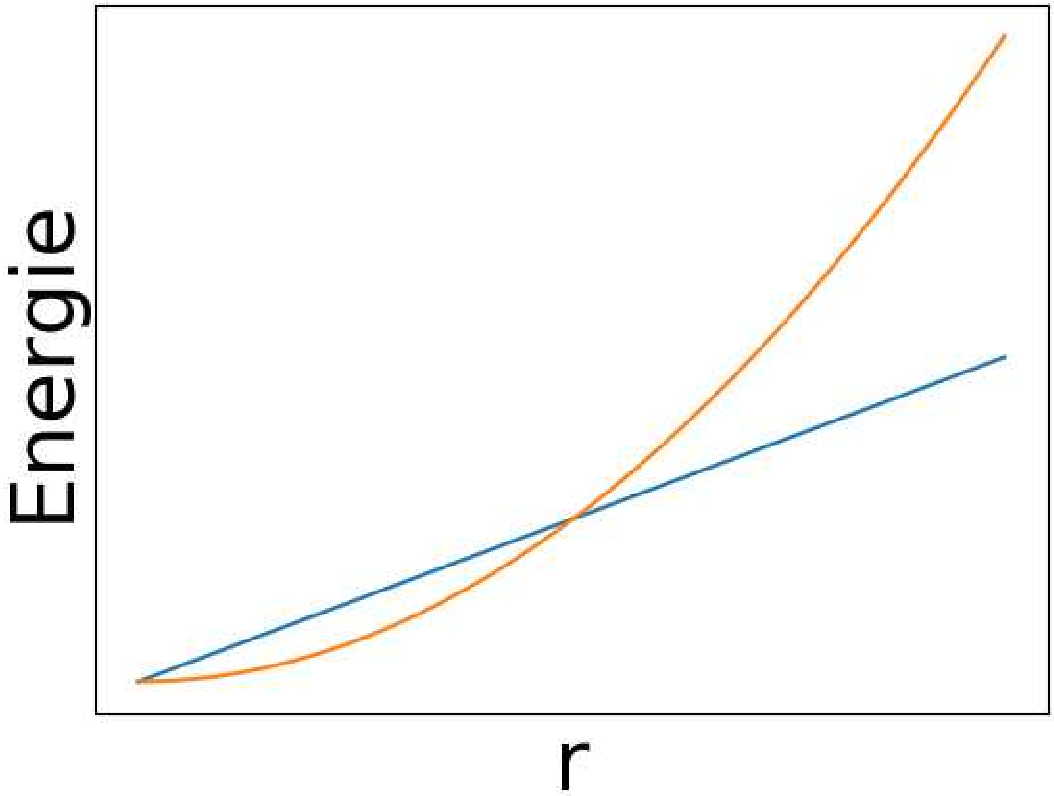
$$2\,E_{\mathrm{kin}} = V^2(R_0) + \frac{1}{2}\left(\frac{\mathrm{dV}}{\mathrm{dR}}\bigg|_{R_0}\right)^2 r^2$$

- Potential energy

$$E_{\mathrm{pot}} = \frac{G\,M_S}{r} = \frac{G\,\Sigma\,\pi\,r^2}{r}$$

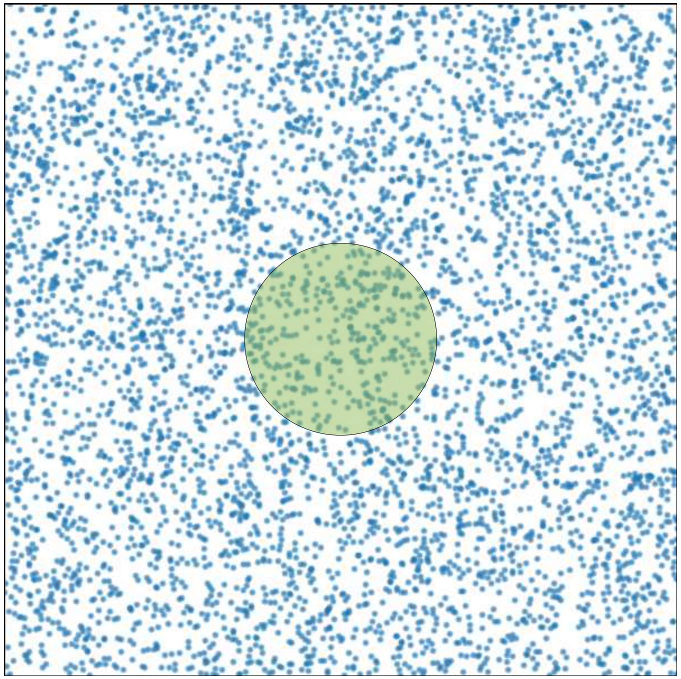
Stabilizing using shearing motions

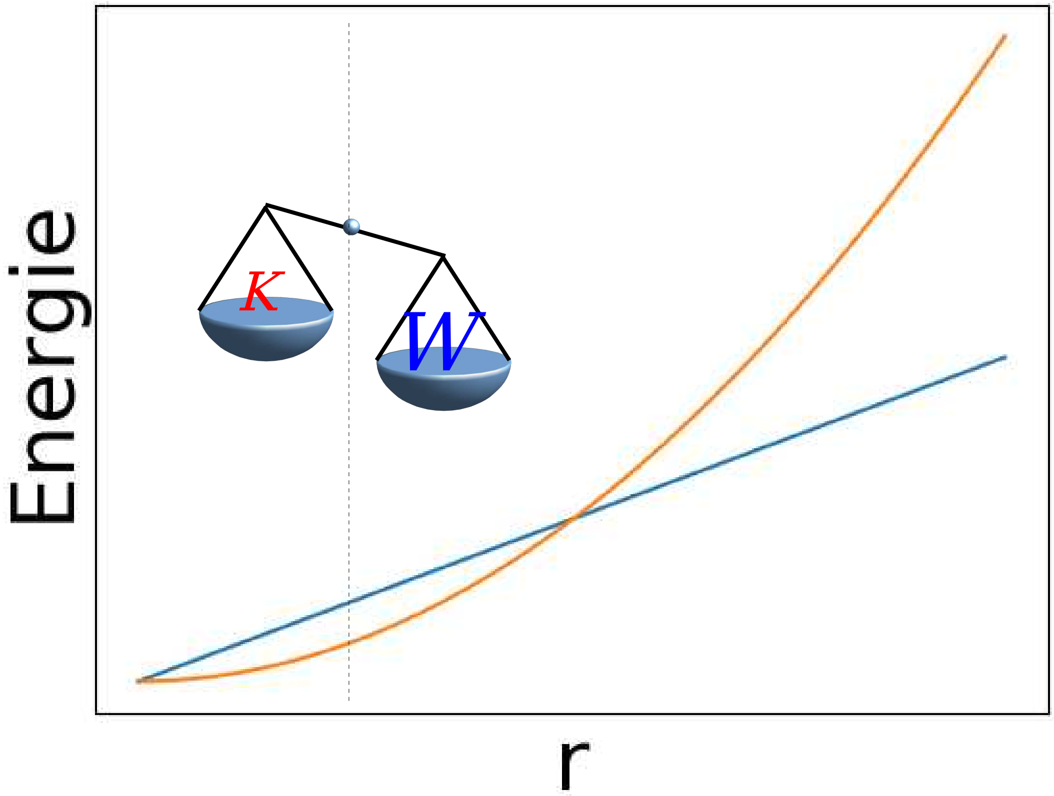




$$K \sim \left(\frac{\mathrm{dV}}{\mathrm{dR}} \bigg|_{R_0} \right)^2 r^2 \qquad W \sim \frac{G\,M(r)}{r} = G\Sigma\pi r$$

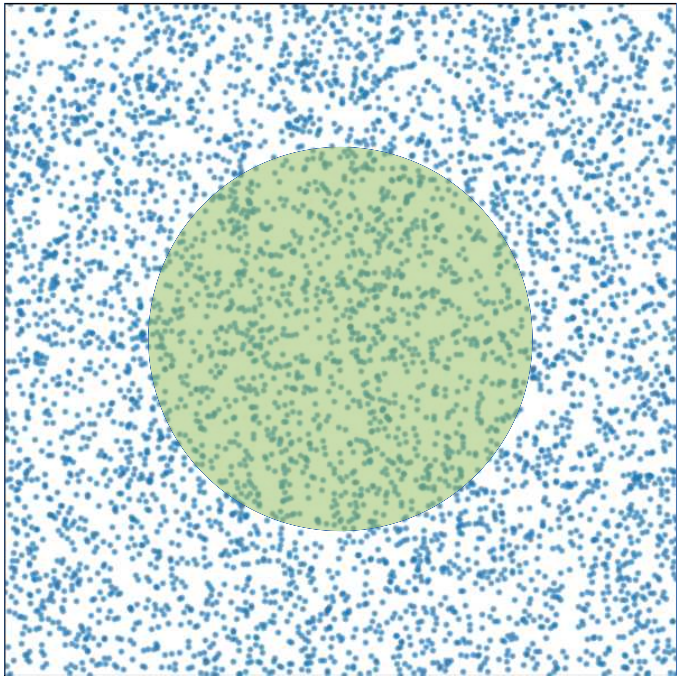
Stabilizing using shearing motions

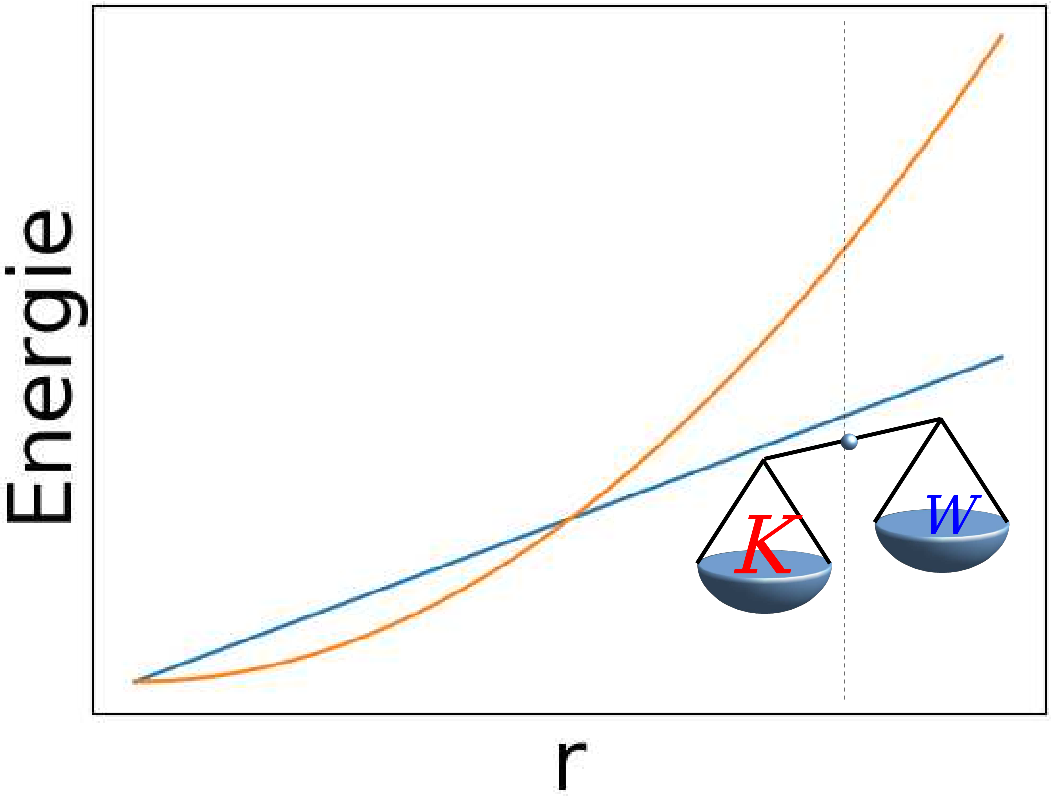




$$K \sim \left(\frac{\mathrm{dV}}{\mathrm{dR}}\bigg|_{R_0} \right)^2 r^2 \qquad W \sim \frac{G\,M(r)}{r} = G\Sigma\pi r$$

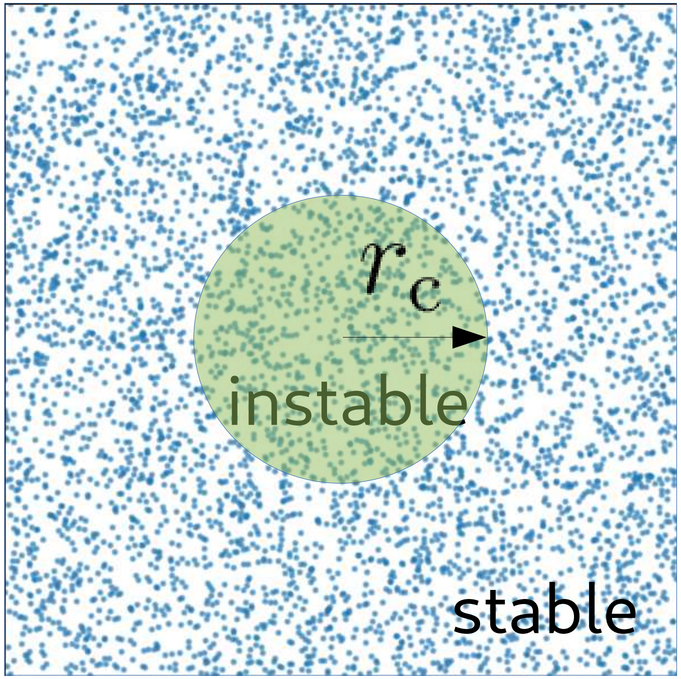
Stabilizing using shearing motions





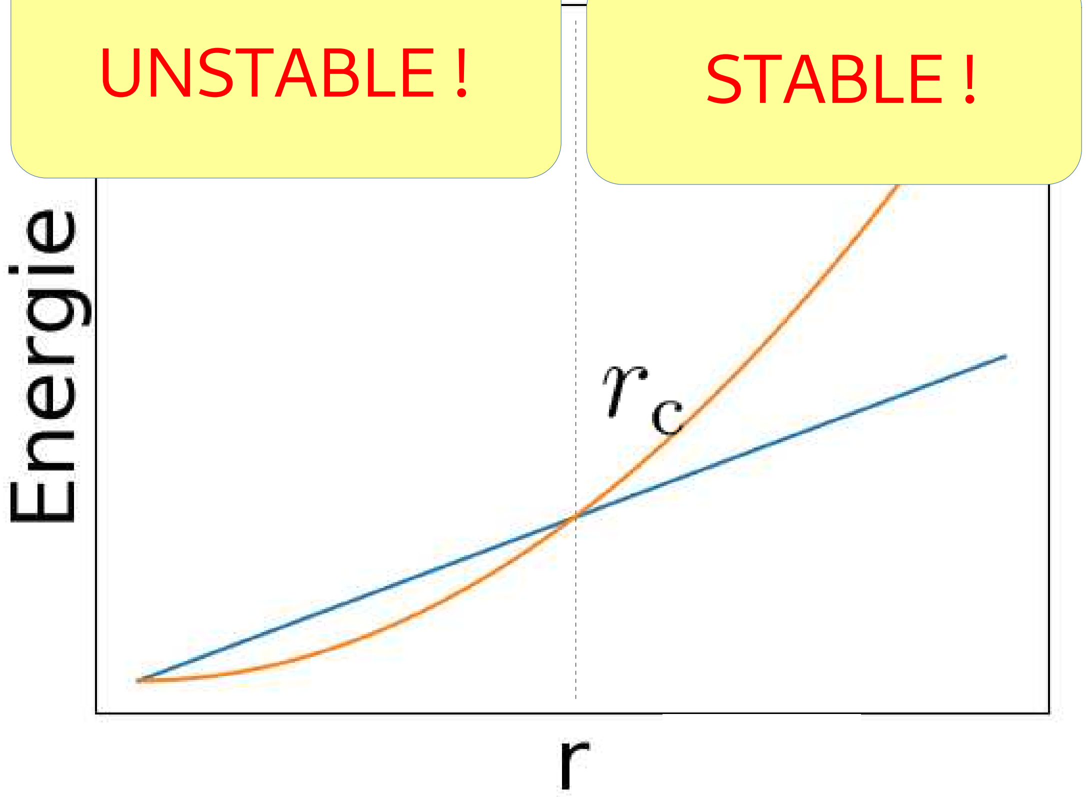
$$K \sim \left(\left. \frac{\mathrm{dV}}{\mathrm{dR}} \right|_{R_0} \right)^2 r^2 \qquad W \sim \frac{G\,M(r)}{r} = G\Sigma\pi r$$

Stabilizing using shearing motions



UNSTABLE !

STABLE !



$$\left(\left.\frac{\mathrm{dV}}{\mathrm{dR}}\right|_{R_0}\right)^2 = 2\pi\,\frac{G\,\Sigma}{r}$$

$$r_c = 2\pi\,\frac{G\,\Sigma}{\left(\left.\frac{\mathrm{dV}}{\mathrm{dR}}\right|_{R_0}\right)^2}$$

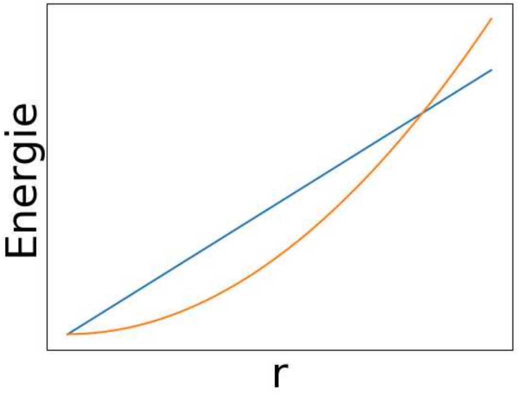
$r > r_c$ stable

$r < r_c$ unstable

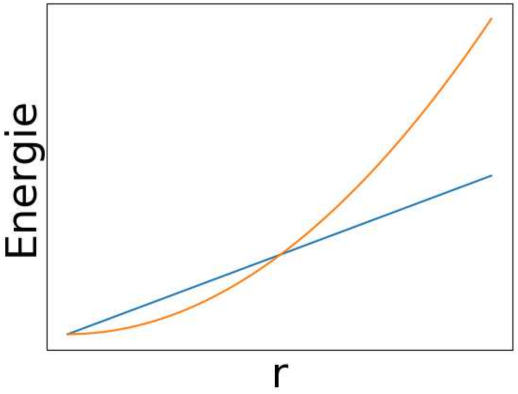
- r_c critical radius
- Beyond this radius, its no longer possible for a perturbation to growth

$V_{\text{vert.}}(R) = \alpha R \qquad \sigma = 0$

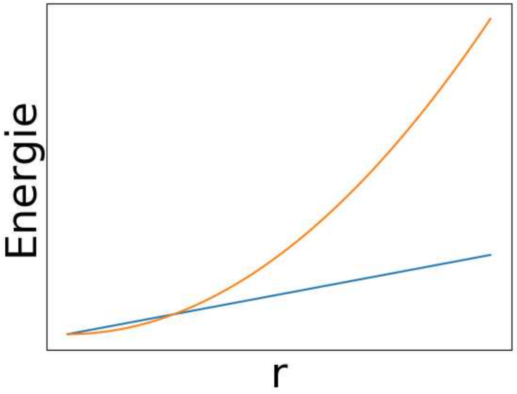
$r_c = 0.25$

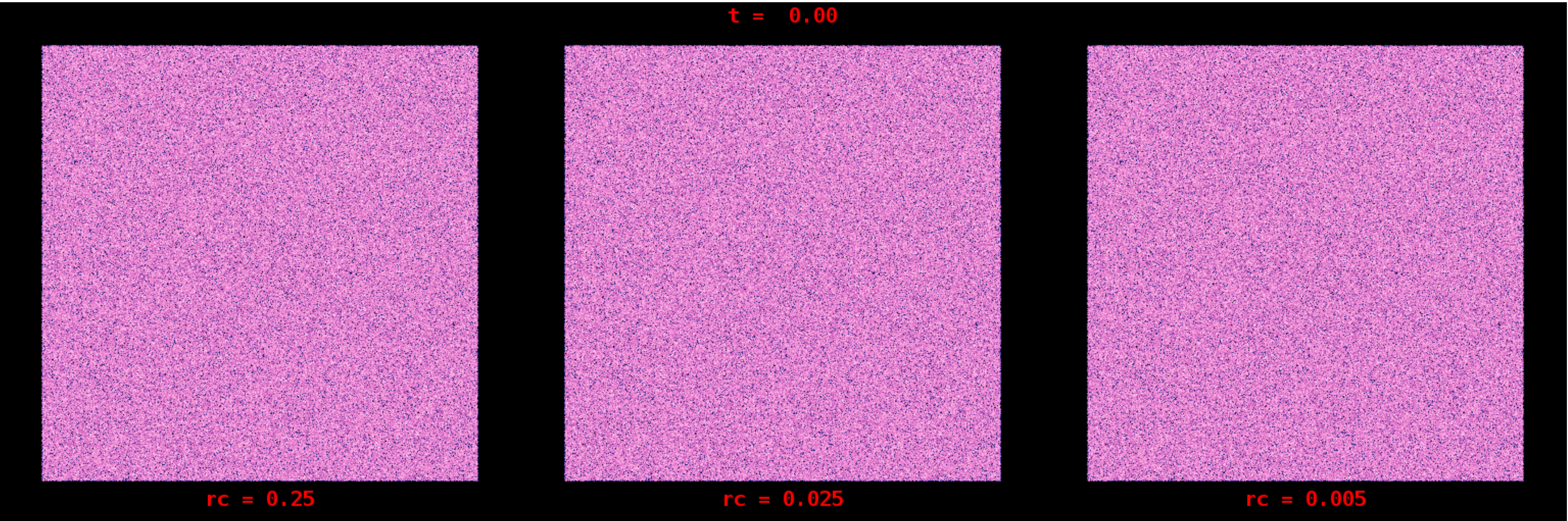


$r_c = 0.025$



$r_c = 0.005$

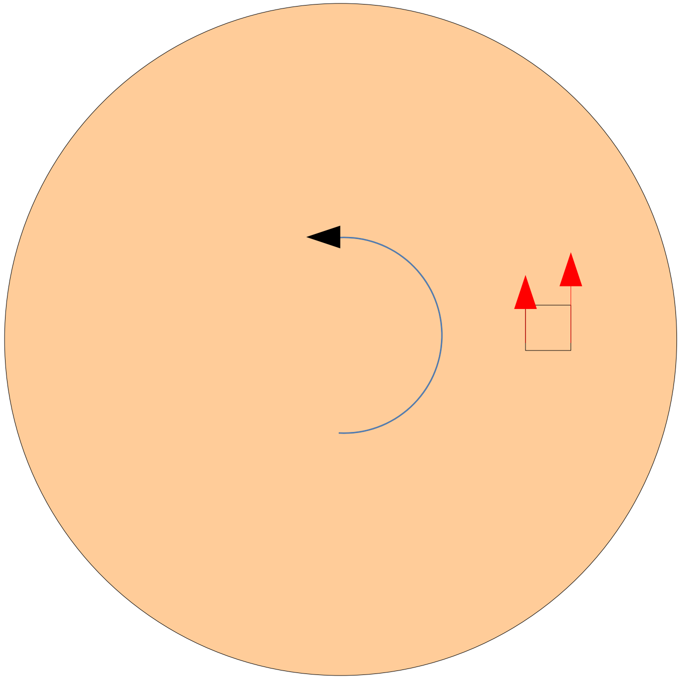




What is the link between slabs and rotating disks ?

Rotating disks of infinite thickness : I - rigid rotation

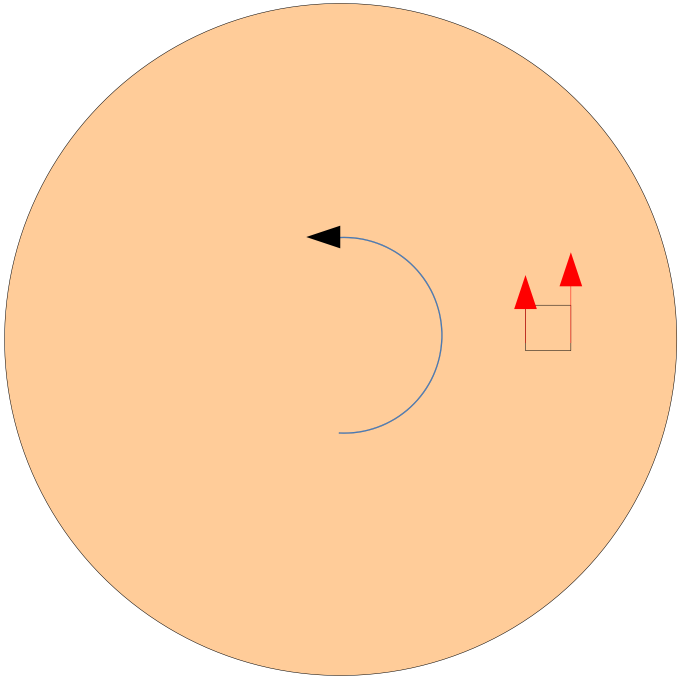
$$\left(\left.\frac{\mathrm{dV}}{\mathrm{dR}}\right|_{R_0}\right)^2 = 2\pi\,\frac{G\,\Sigma}{r} \qquad \left.\frac{\mathrm{dV}}{\mathrm{dR}}\right|_{R_0} = \Omega_0$$



$$V = \Omega \cdot r$$

Rotating disks of infinite thickness : I - rigid rotation

$$\left(\left.\frac{\mathrm{dV}}{\mathrm{dR}}\right|_{R_0} \right)^2 = 2\pi \frac{G\,\Sigma}{r} \qquad \left.\frac{\mathrm{dV}}{\mathrm{dR}}\right|_{R_0} = \Omega_0$$



$$\Omega_0^2 = 2\pi \frac{G\,\Sigma}{r}$$

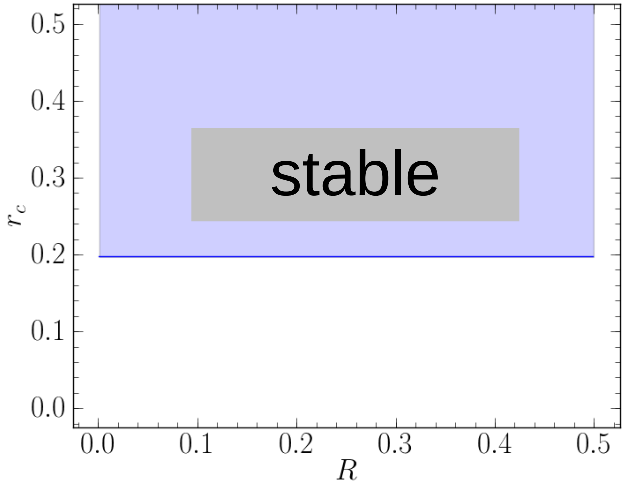
$$\frac{1}{2}\Omega_0^2\, r^2 = \frac{G\,M}{r}$$

Critical radius

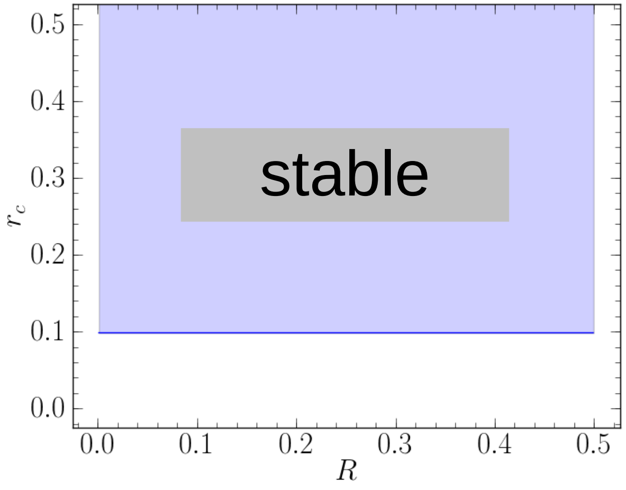
$$r_c = 2\pi \frac{G\,\Sigma}{\Omega_0^2}$$

Rotating disks of infinite thickness : I - rigid rotation

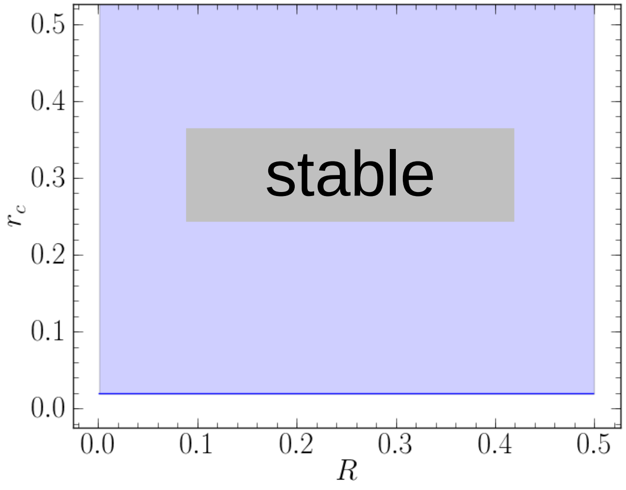
$r_c = 0.25, \sigma = 0$

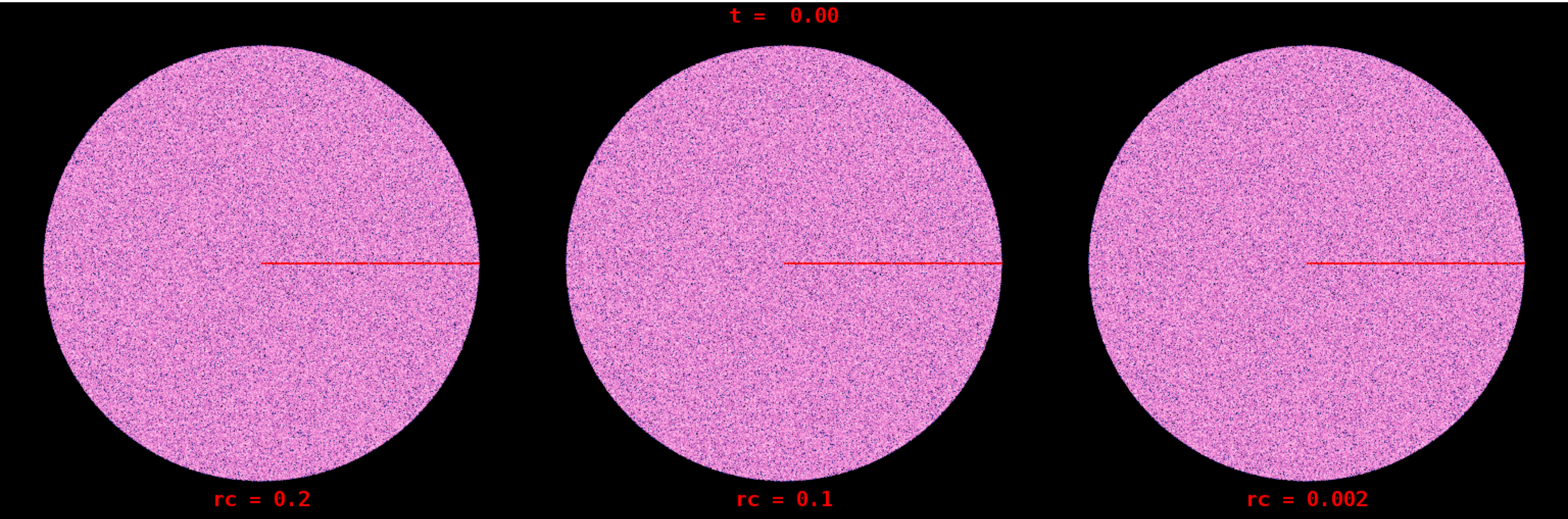


$r_c = 0.025, \sigma = 0$



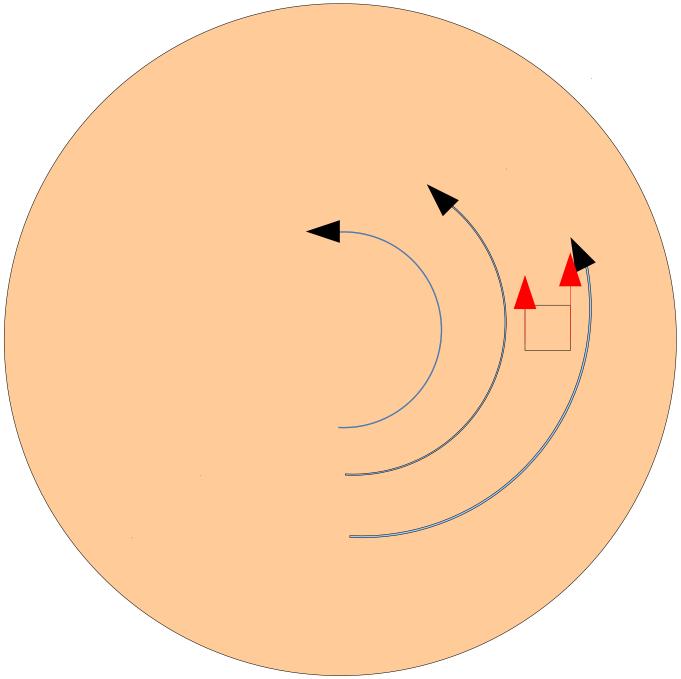
$r_c = 0.005, \sigma = 0$





Rotating disks of infinite thickness : II - differential rotation

Need to develop the velocity up to the second order



Rotating disks of infinite thickness : II - differential rotation

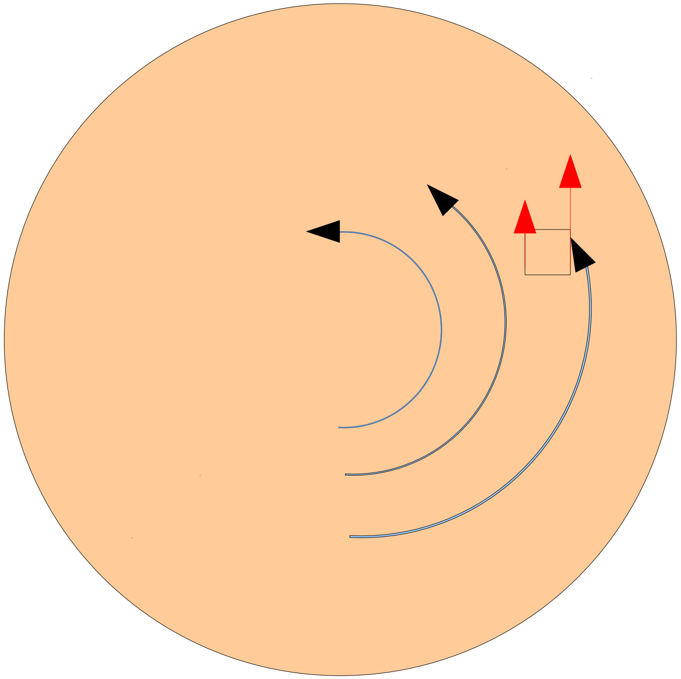
Need to develop the velocity up to the second order

$$\kappa^2 = 8\pi \frac{G\Sigma}{r}$$

$$\frac{1}{8}\kappa^2 r^2 = \frac{G\,M}{r}$$

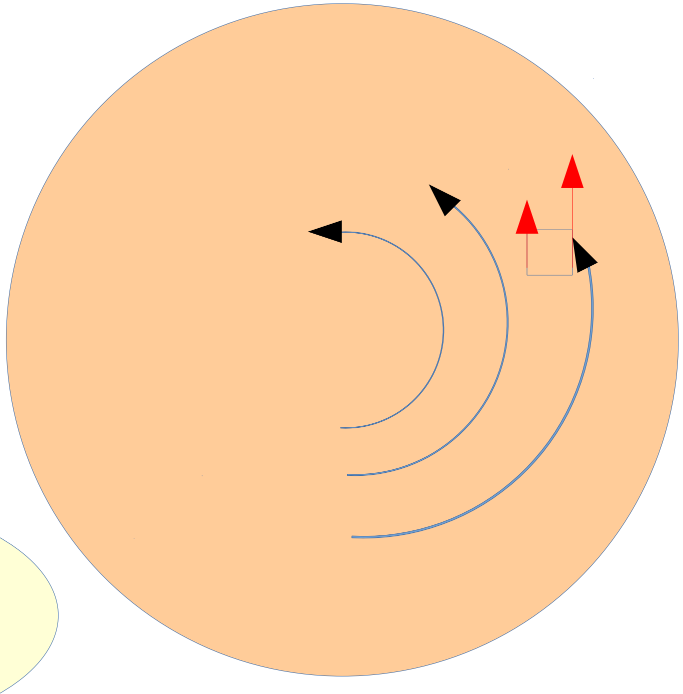
Critical radius

$$r_c = 8\pi \frac{G\Sigma}{\kappa^2}$$



Rotating disks of infinite thickness : II - differential rotation

Need to develop the velocity up to the second order



$$\kappa^2 = 8\pi \frac{G\Sigma}{r}$$

$$\frac{1}{8}\kappa^2 r^2 = \frac{G\,M}{r}$$

Critical radius

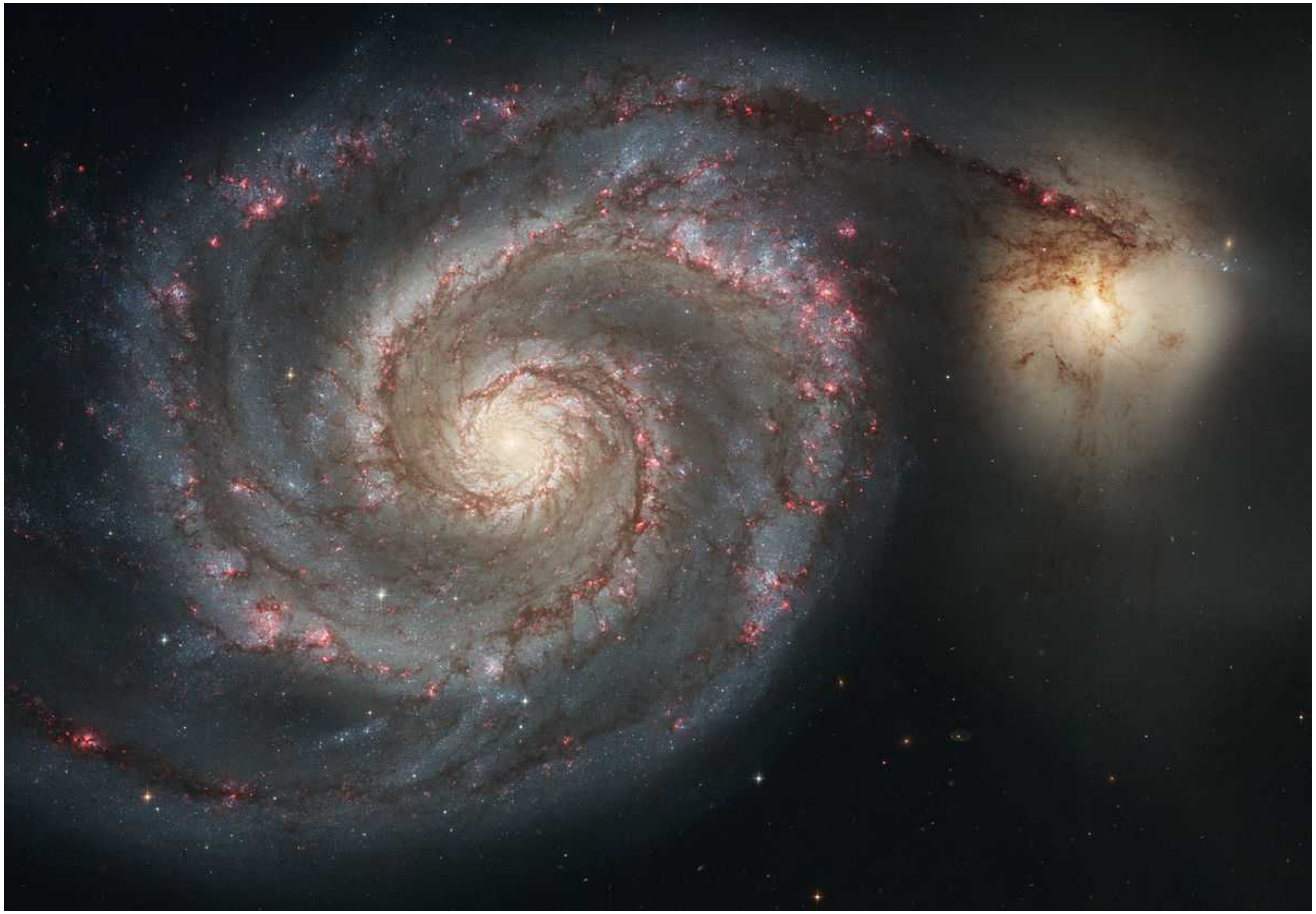
$$r_c = 8\pi \frac{G\Sigma}{\kappa^2}$$

Classical derivation

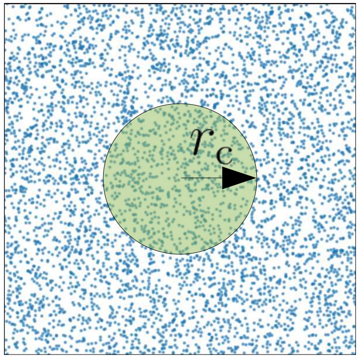
$$\omega^2 = \kappa^2 - 2\pi G\Sigma |k|$$

$$r_c = \frac{\lambda_c}{2} = 2\pi^2 \frac{G\Sigma}{\kappa^2}$$

Predicting the number of spiral arms...



Rotating disks of infinite thickness : the number of spiral arms



- For a rotating disk of a given surf. density Σ

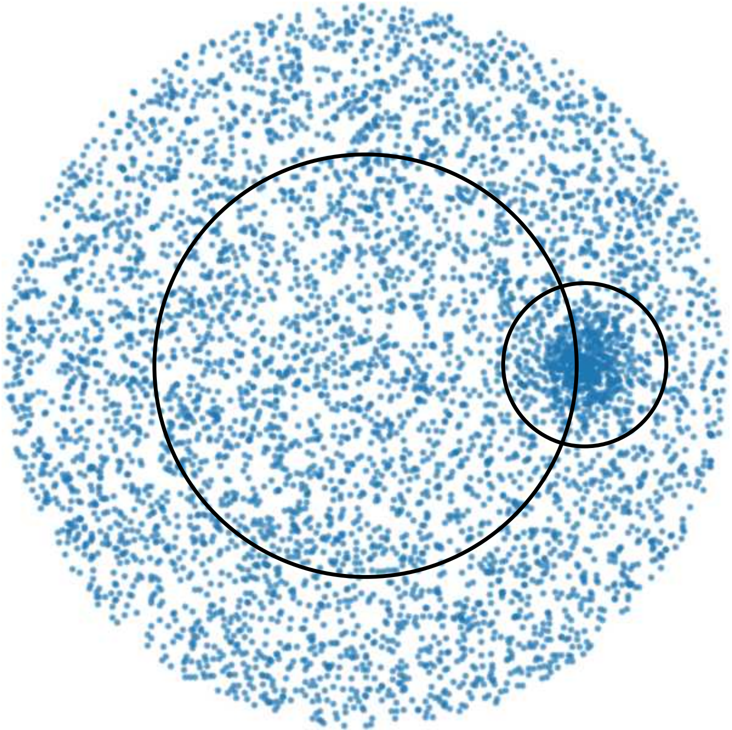
  the radial epicycle frequency κ

  determines the maximal size

  of the clumps

$$r_c = 8\pi \frac{G\,\Sigma}{\kappa^2}$$

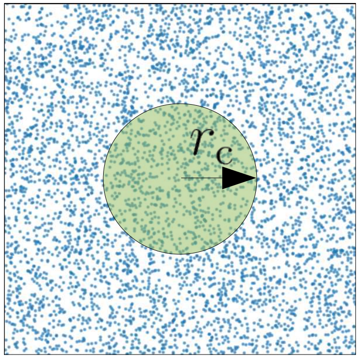
- Number of clumps per radius



$$N_\mathrm{c} = \frac{L}{2\,r_c} = \frac{\kappa^2\,R}{8\,G\,\Sigma}$$

$$L = 2\pi\,R$$

Rotating disks of infinite thickness : the number of spiral arms



- For a rotating disk of a given surf. density Σ

  the radial epicycle frequency κ

  determines the maximal size

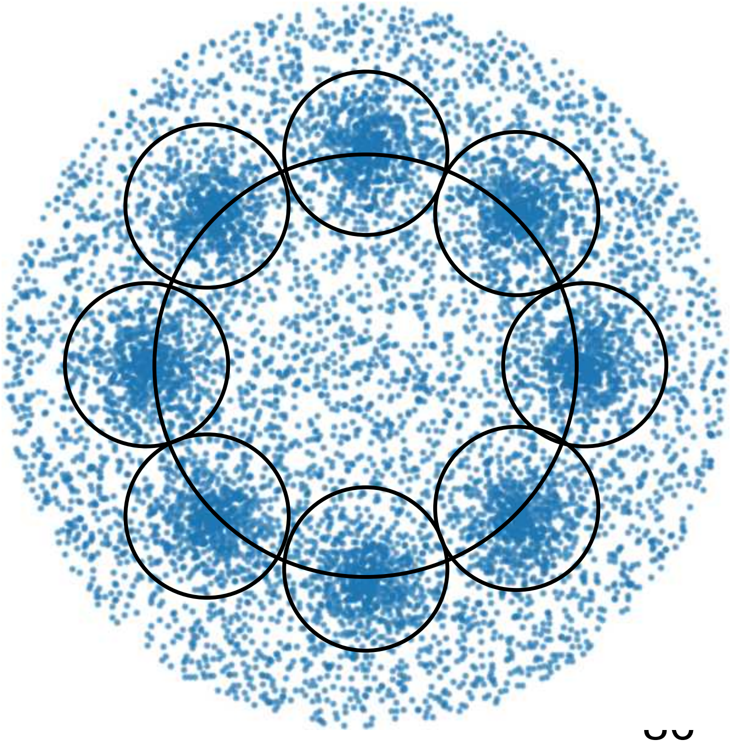
  of the clumps

$$r_c = 8\pi \frac{G\,\Sigma}{\kappa^2}$$

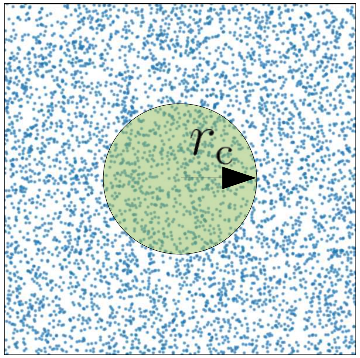
- Number of clumps per radius

$$N_{\rm c} = \frac{L}{2\,r_c} = \frac{\kappa^2\,R}{8\,G\,\Sigma}$$

$$L = 2\pi\,R$$



Rotating disks of infinite thickness : the number of spiral arms



- For a rotating disk of a given surf. density Σ

  the radial epicycle frequency κ

  determines the maximal size

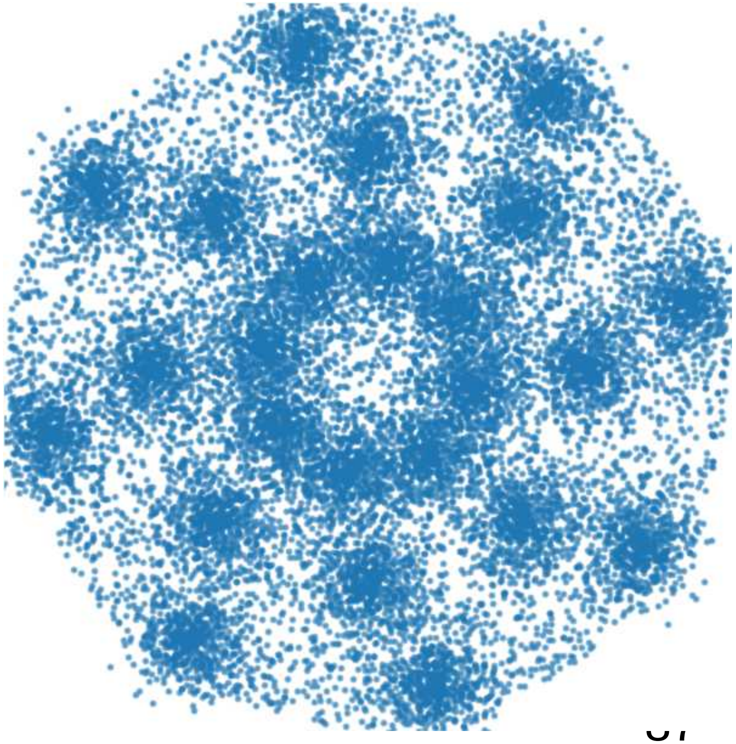
  of the clumps

$$r_c = 8\pi \frac{G\,\Sigma}{\kappa^2}$$

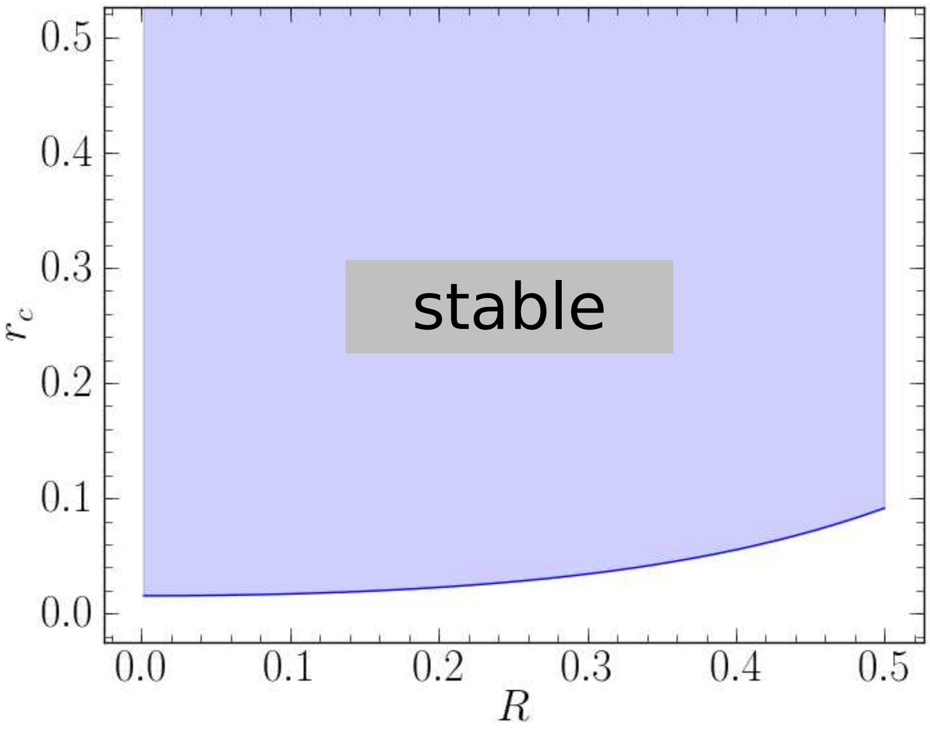
- Number of clumps per radius

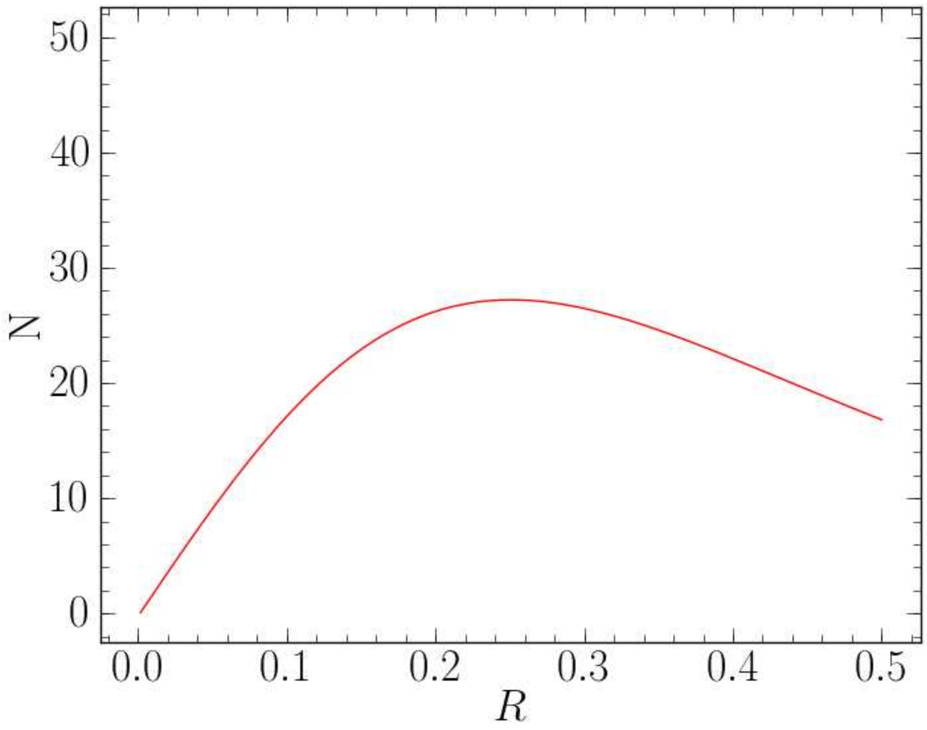
$$N_c = \frac{L}{2\,r_c} = \frac{\kappa^2\,R}{8\,G\,\Sigma}$$

$$L = 2\pi\,R$$

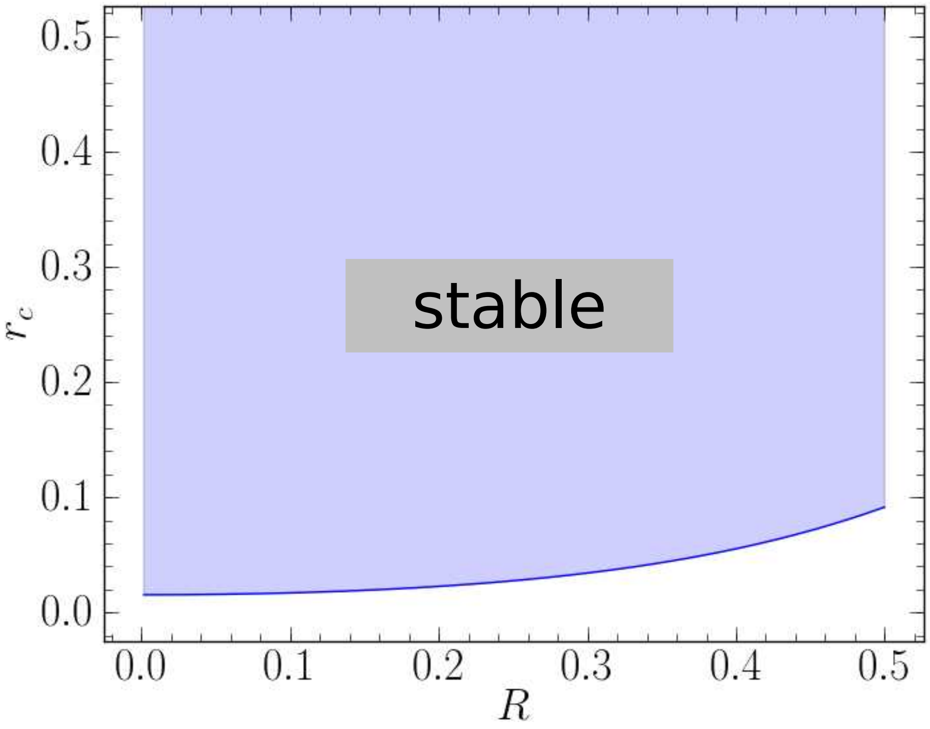


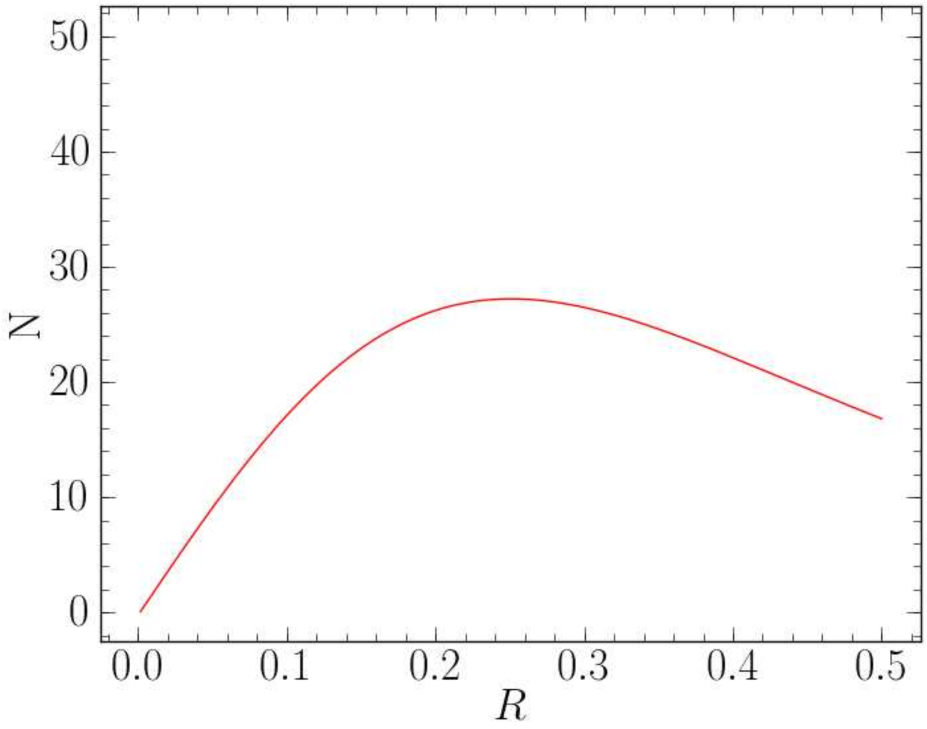
Rotating disks of infinite thickness : the number of spiral arms





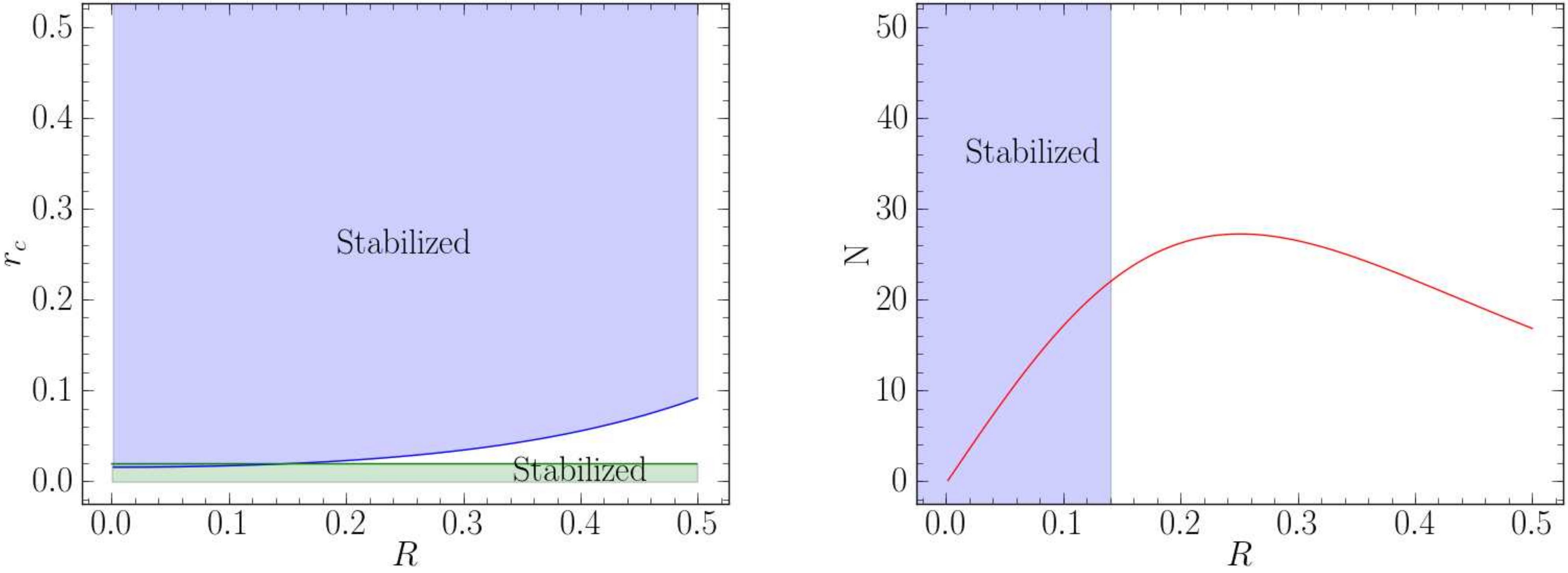
Rotating disks of infinite thickness : the number of spiral arms



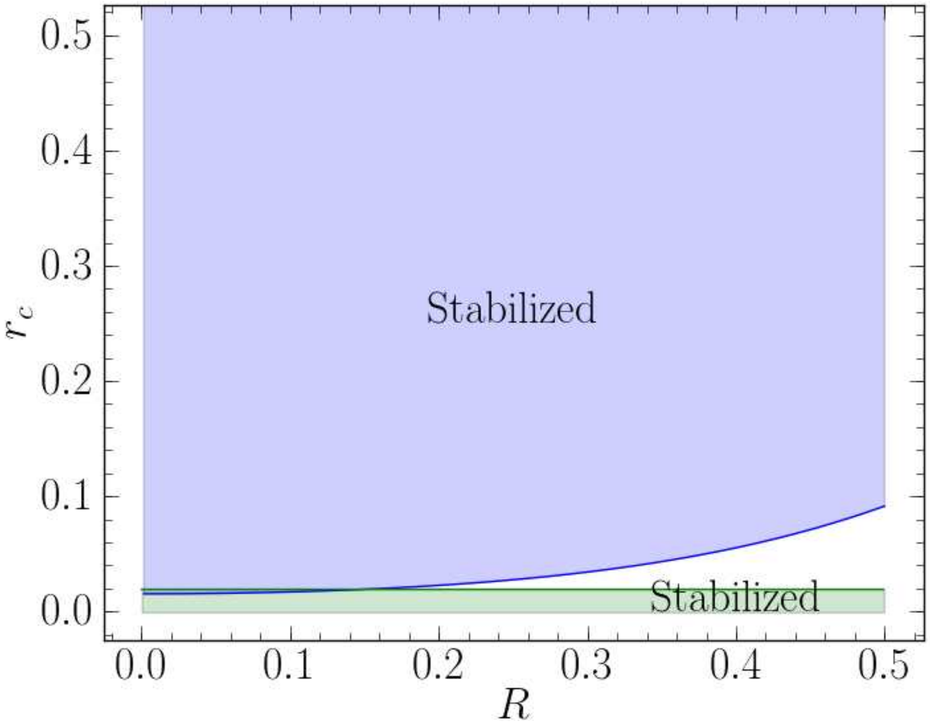


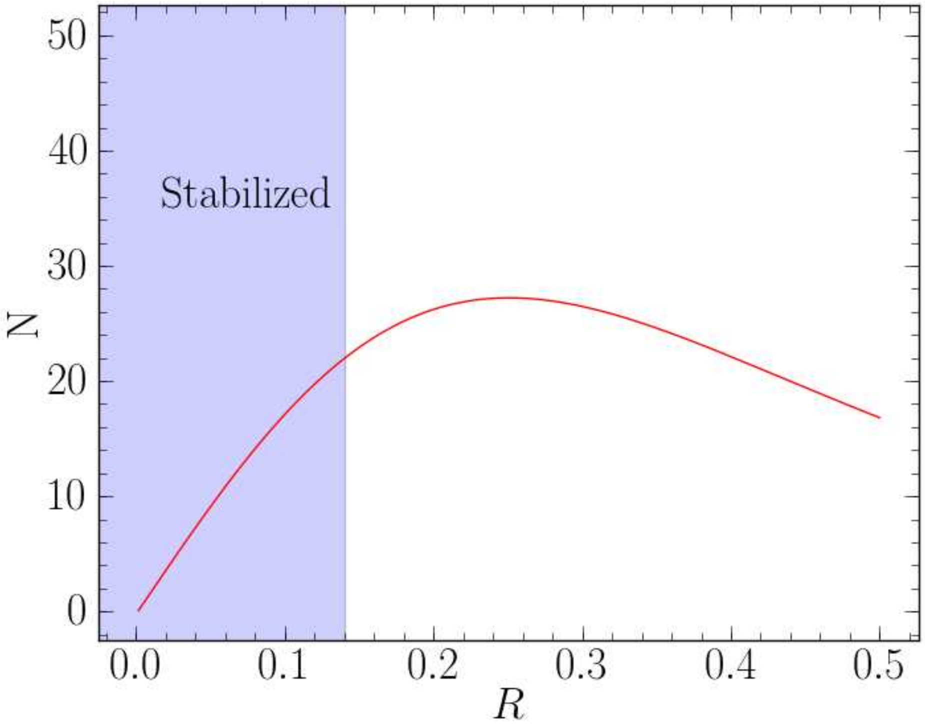
Putting all together...

Rotating disks of infinite thickness : adding velocity dispersion

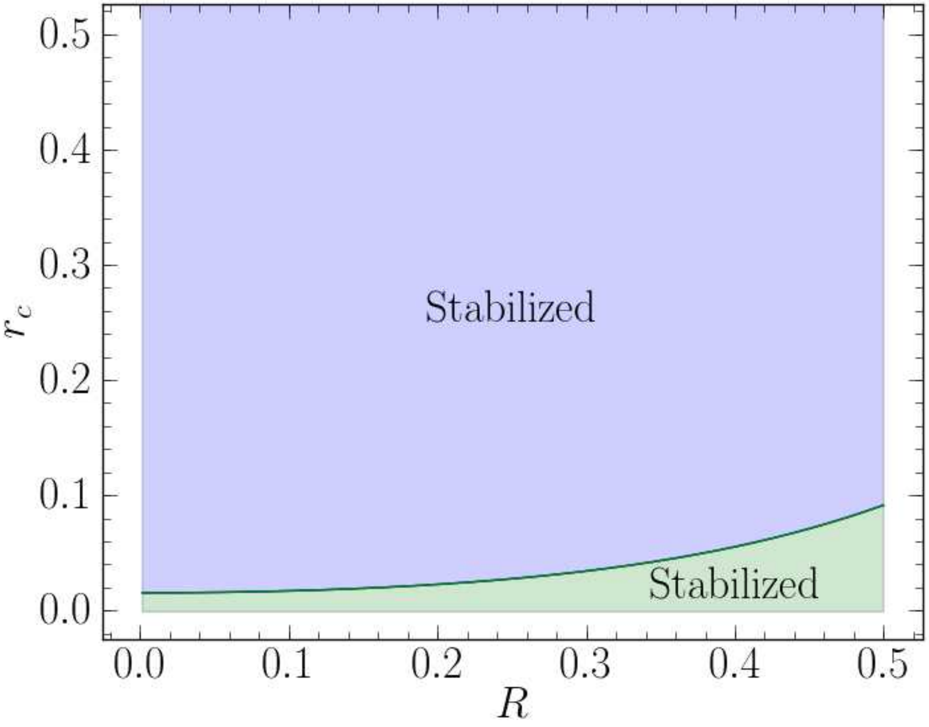


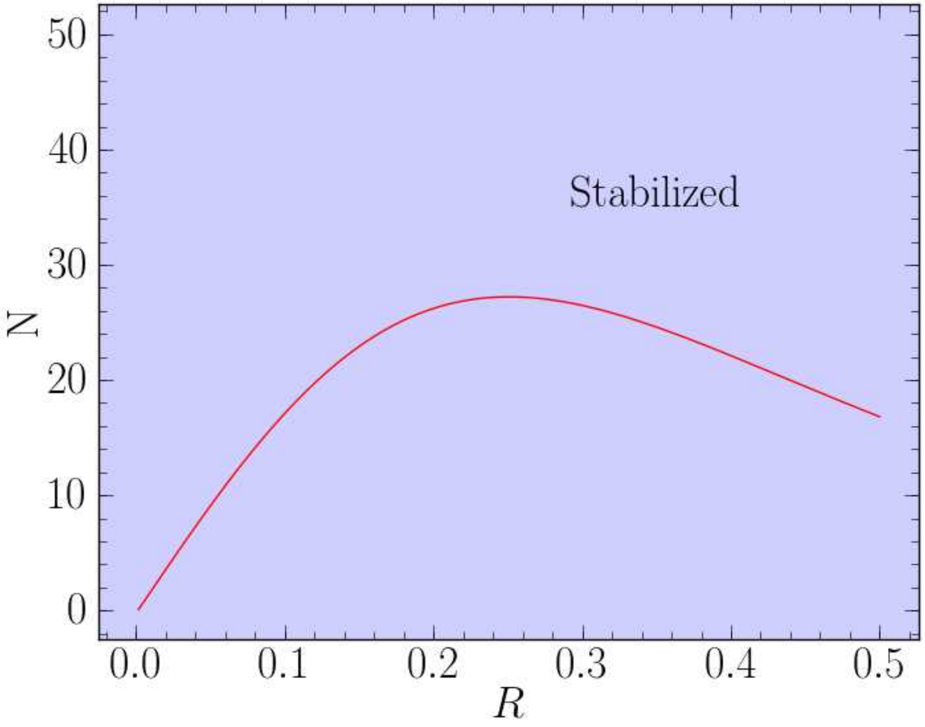
Rotating disks of infinite thickness : adding velocity dispersion



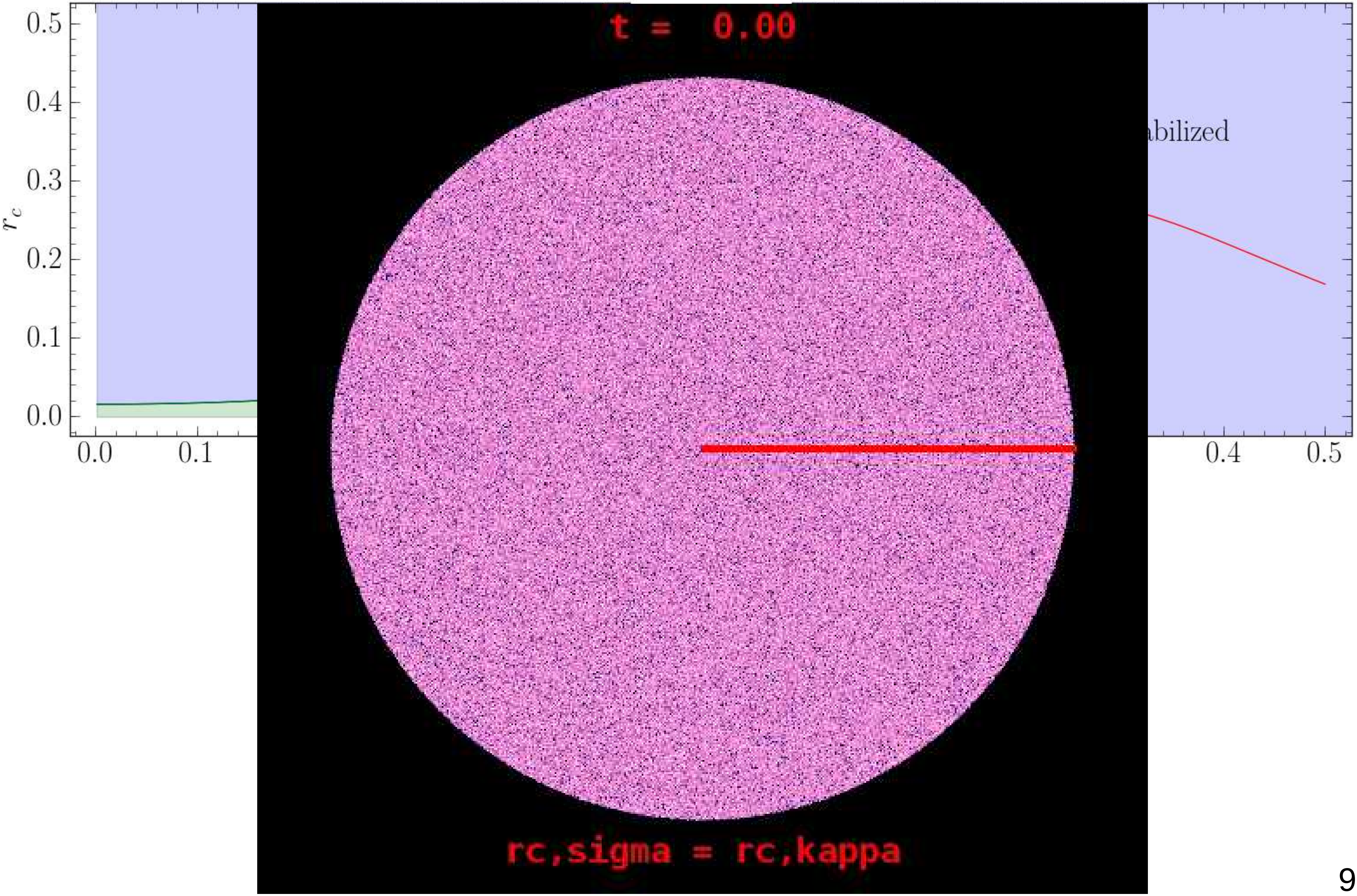


Rotating disks of infinite thickness : Local stability

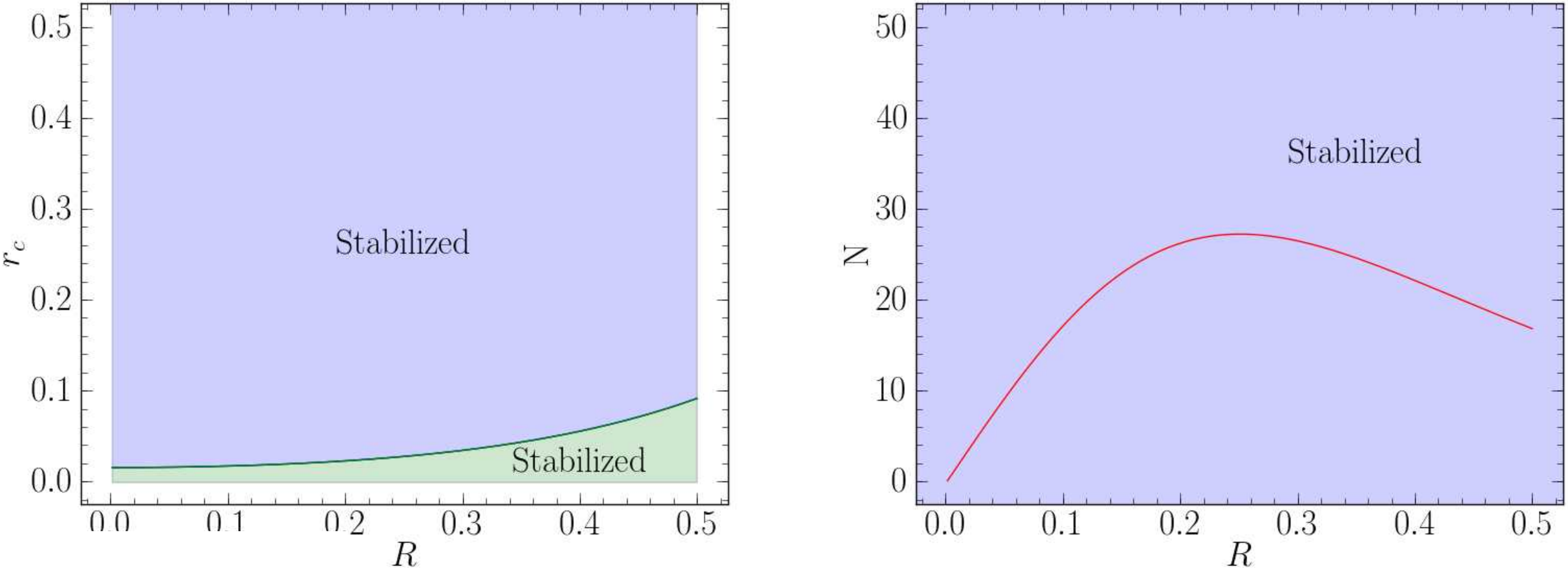




Rotating disks of infinite thickness : Local stability



Rotating disks of infinite thickness : Local stability

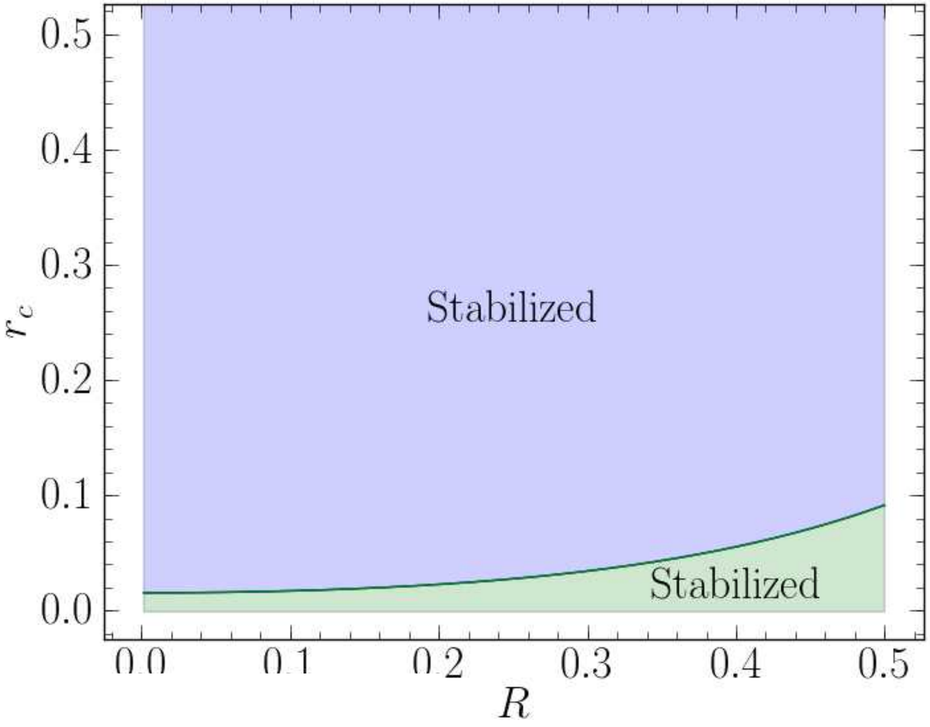


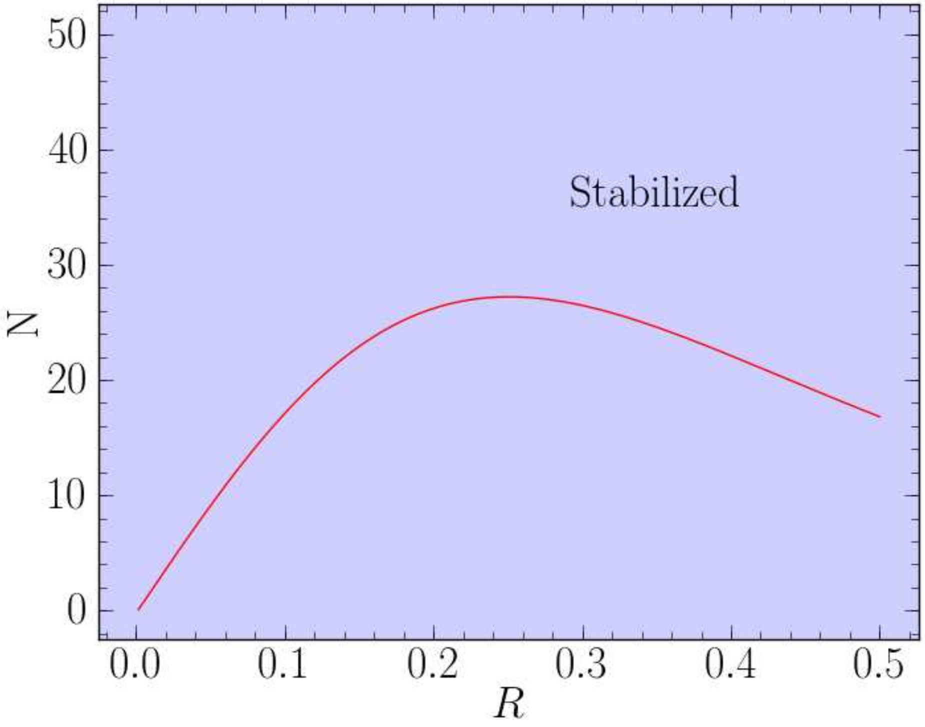
$$\sigma^2 > \frac{G\,M}{r}$$

$$\frac{1}{8}\kappa^2\,r^2 > \frac{G\,M}{r}$$

$$\sigma\kappa\,r > \sqrt{8}\frac{G\,M}{r} \longrightarrow \boxed{\frac{\sigma\,\kappa}{\pi\,G\,\Sigma} > \sqrt{8}}$$

Rotating disks of infinite thickness : Local stability





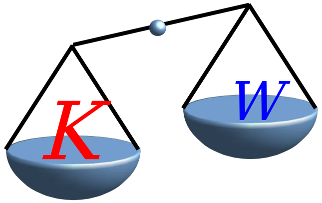
$$\begin{tikzcd} \sigma^2 > \frac{G\,M}{r} \arrow[dr] & & \\ & \sigma\kappa\, r > \sqrt{8}\frac{G\,M}{r} \arrow[r] & \boxed{\frac{\sigma\,\kappa}{\pi\,G\,\Sigma} > \sqrt{8}} \\ \frac{1}{8}\kappa^2\, r^2 > \frac{G\,M}{r} \arrow[ur] & & \end{tikzcd}$$

Safronov-Toomre critterion

(Safronov 1960, Toomre 1964)

$$Q = \frac{\sigma_R\,\kappa}{3.36\,G\,\Sigma};$$

Disk stability : summary



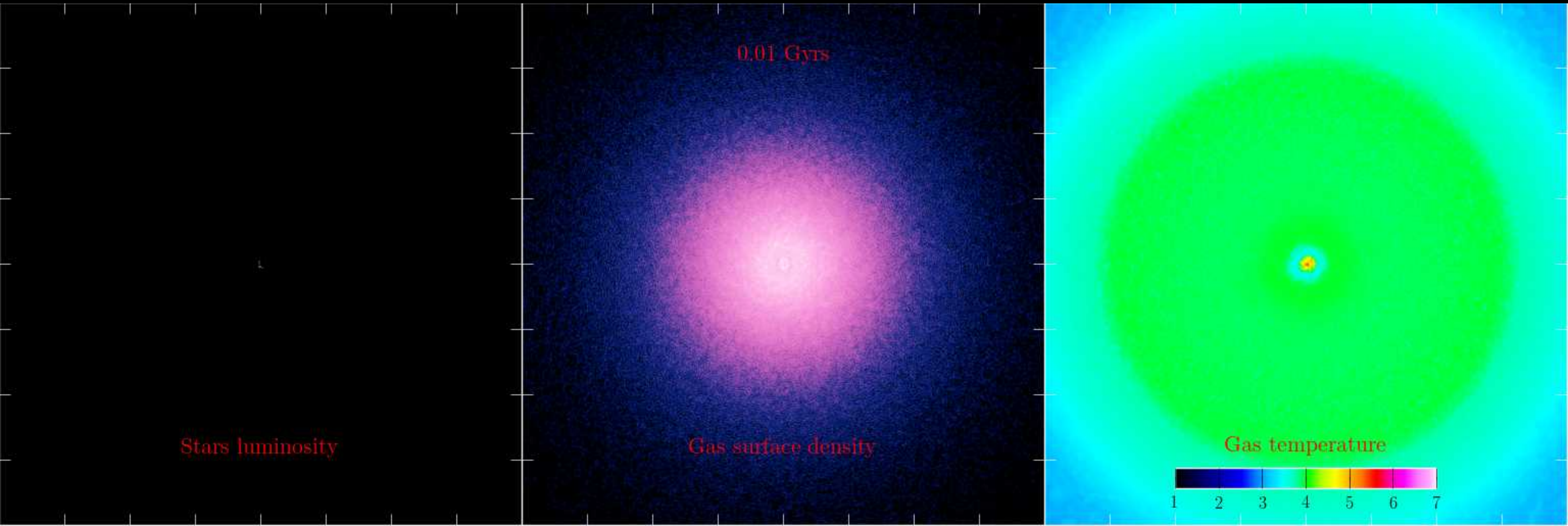
1. large random motions : σ

→ stabilizes the small scales $\sigma^2 > G\,\Sigma\,\pi\,r$

2. strong differential rotation : κ

→ stabilizes the large scales $\kappa^2\,r^2 > G\,\Sigma\,\pi\,r$

Realistic Simulations of galactic disks

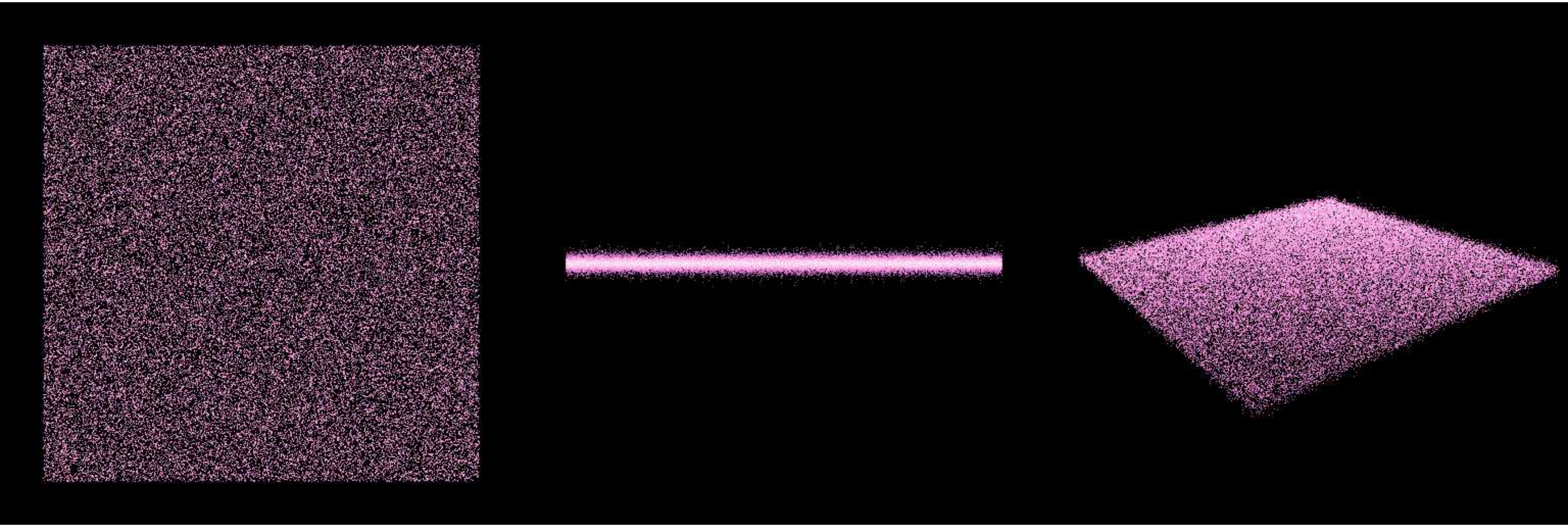


M74

Nature is always more tricky…

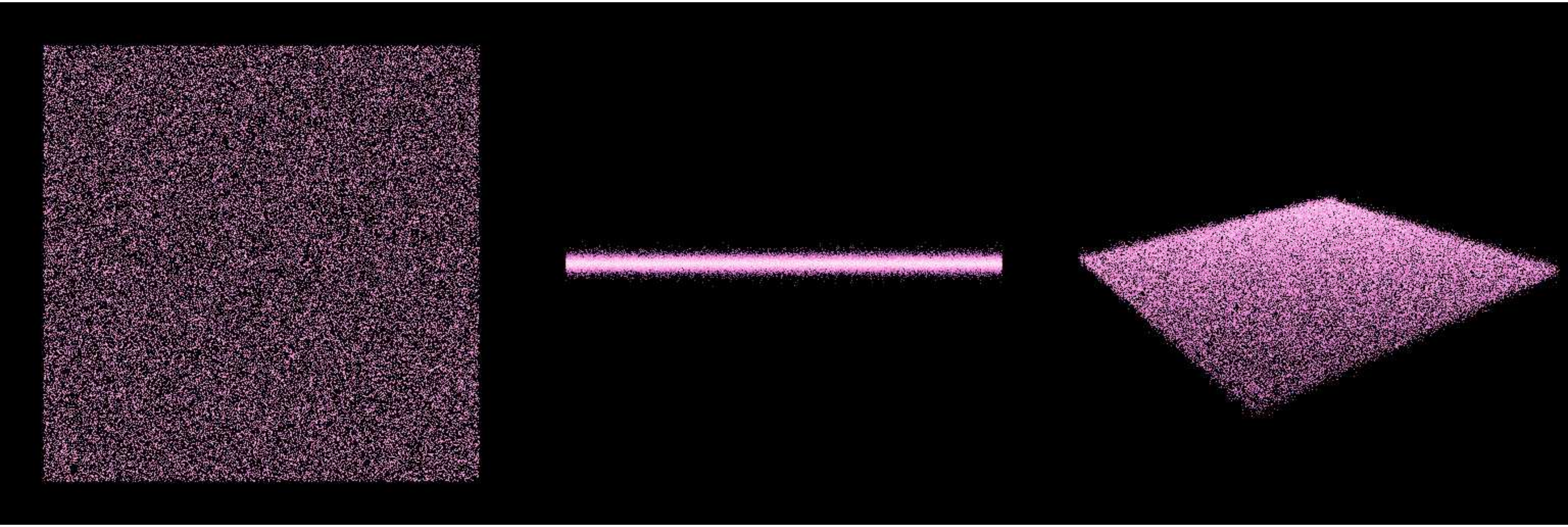
The stability of infinite slab of finite thickness

- isothermal vertical profile



The stability of infinite slab of finite thickness

- isothermal vertical profile



Dispersion relations

horizontal perturbation

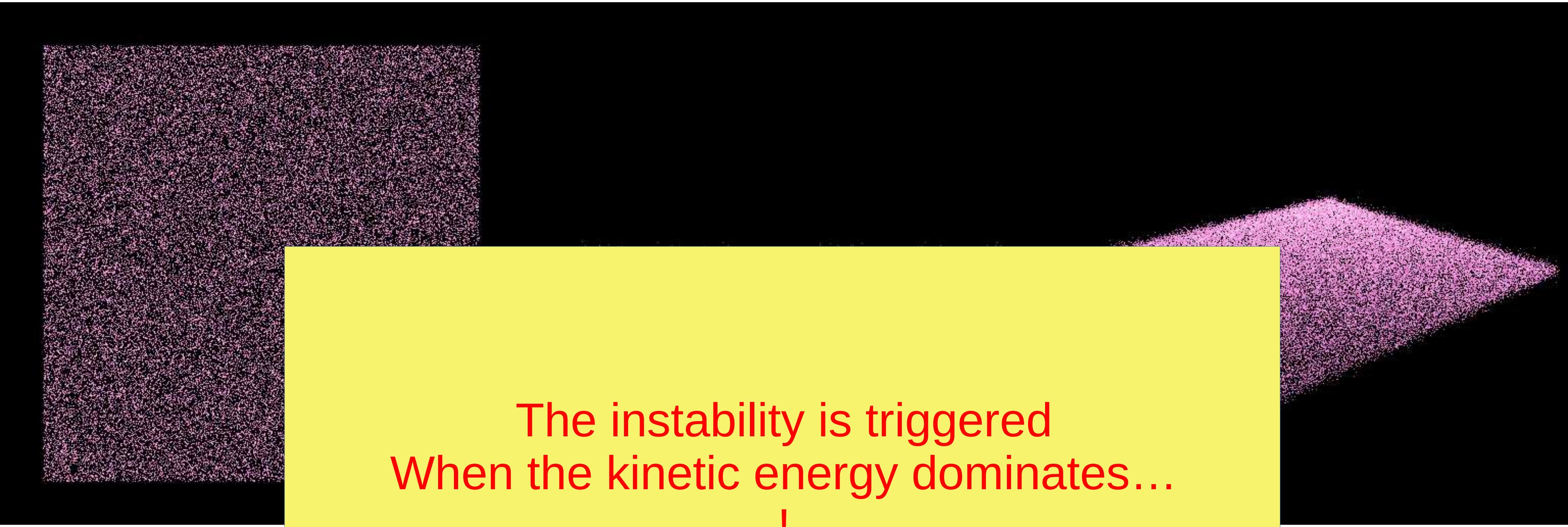
$$\omega^2 = \kappa^2 - 2\pi G\Sigma k + c_s^2 k^2$$

vertical perturbation

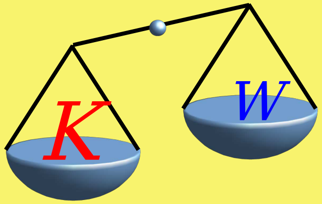
$$\omega^2 = \underset{\uparrow}{\nu^2} \underset{\uparrow}{+} 2\pi G\Sigma k \underset{\uparrow}{-} \sigma^2 k^2$$

The stability of infinite slab of finite thickness

- isothermal vertical profile



The instability is triggered
When the kinetic energy dominates…
!



Dispersion re

horizontal pe

vertical pertu

Merry Christmas
and
happy new year 2022 !

JORGE CHAM © 2003
www.phdcomics.com

The End

Stability of collisionless systems

The stability of uniformly rotating systems

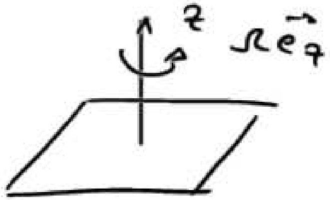
(additional material)

The stability of uniformely rotating systems

- Flattened systems : the geometry is more complex
- Reservoir of kinetic energy (rotation) to feed unstable modes

The uniformly rotating sheet

- infinite disk of zero thickness with surface density Σ_0
    - plane $z = 0$
    - rotation $\vec{\Omega} = \Omega \vec{e_z}$



- 2D perturbation / evolution (no warp, no bending)

① Continuity equation

(Fluid) $\Sigma_d(x,y,t)$, $\phi(x,y,t)$, $v(x,y,t)$, $p(x,y,t)$

6 inc
6 Equ.

$$\frac{\partial \rho_s}{\partial t} + \vec{\nabla}(\rho_s \vec{v}) = 0 \longrightarrow \boxed{\frac{\partial \Sigma_d}{\partial t} + \vec{\nabla}(\Sigma_d \vec{v}) = 0}$$

with $\vec{v} = \vec{v}(x,y,t) = v_x(x,y,t)\,\vec{e_x} + v_y(x,y,t)\,\vec{e_y}$ (in the plane)

② Euler Equation

$$\frac{\partial}{\partial t}\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\rho_s} - \vec{\nabla}\phi$$

$$\rightarrow \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\Sigma_d} - \vec{\nabla}\phi \overbrace{- 2\vec{\Omega}\times\vec{v}}^{\text{Coriolis force}} + \overbrace{-\vec{\Omega}\times(\vec{\Omega}\times\vec{x})}^{\text{centrifugal force}}$$

$$\boxed{\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\Sigma_d} - \vec{\nabla}\phi - 2\Omega(v_x\vec{e_x} + v_y\vec{e_y}) + \Omega^2(x\vec{e_x} + y\vec{e_y})}$$

③ Poisson equation

$$\nabla^2 \phi = 4\pi G \rho \longrightarrow \boxed{\nabla^2 \phi = 4\pi G \Sigma \, \delta(z)}$$

④ Equation of state (barotropic)

$$p(x, y, t) = p(\Sigma_d(x, y, t))$$

Notes

- sound speed : $v_s^2 = \left.\frac{\partial P(\Sigma)}{\partial \Sigma}\right|_{\Sigma_0}$
- P is the pressure in the plane only $\left[\frac{\text{force}}{\text{lenght}} \right]$
- Σ_d : surf density of the disk only

Σ = total surface density $\quad \Sigma_d + \underset{\text{External perturbation}}{\underset{\sim}{\Sigma_e}}$

Isolated system at equilibrium Σ_{d0}, ϕ_0, p_0 , $\vec{v}_0 = 0$

① Continuity equation $\rightarrow$ $0 = 0$

② Euler Equation $\vec{\nabla}\phi_0 = \Omega^2(x\vec{e}_x + y\vec{e}_y)$

F_{grav} $F_{cent.}$

③ Poisson equation $\nabla^2\phi_0 = 4\pi G \Sigma_{d0}\,\delta(z)$ $(\vec{\nabla}p_0 = 0)$

Note by symmetry, $\vec{\nabla}\phi_0 \parallel \vec{e}_z$

$$\Rightarrow \quad \phi_0 = 2\pi G \Sigma_0 |z|$$

poisson $\Rightarrow$ $\Sigma_{d0} = 0$

(need a Jeans swindle

consider only Poisson

for the perturbed part)

The response of the system to a weak perturbation $-\varepsilon \vec{\nabla} \phi_e$

$$\Sigma_{d0} \to \Sigma_{d0} + \varepsilon \Sigma_{d1}(x, y, t)$$

$$\vec{v}_0 = 0 \to \varepsilon v_1(x, y, t)$$

$$\Sigma_0 \to \Sigma_0 + \varepsilon \Sigma_1(x, y, t)$$ total density

$$\phi_0 \to \phi_0 + \varepsilon \phi_1(x, y, t)$$ total potential

$$= \varepsilon \phi_{d1}(x, y, t) + \varepsilon \phi_e(x, y, t)$$

A first order in ε, we get

① Continuity equation

$$\frac{\partial}{\partial t} \Sigma_{d1} + \Sigma_0 \vec{\nabla} \vec{v}_1 = 0$$

② Euler Equation

$$\frac{\partial}{\partial t} \vec{v}_1 = -\frac{v_s^2}{\Sigma_0} \vec{\nabla} \Sigma_{d1} - \vec{\nabla} \phi_1 - \underbrace{2 \vec{\Omega} \times \vec{v}_1}_{\text{contribution of the rotation}}$$

③ Poisson equation

$$\nabla^2 \phi_1 = 4\pi G \Sigma_1 \delta(z)$$

④ "Equation of state"

$$v_s^2 = \left.\frac{\partial p}{\partial \Sigma}\right|_{\Sigma_0}$$

the rest is similar to the homogeneous case

Solution

$$\frac{\partial}{\partial t} \text{①} \quad : \quad \frac{\partial^2}{\partial t^2} \Sigma_{d1} + \Sigma_0 \vec{\nabla} \frac{\partial}{\partial t} \vec{v}_1 = 0$$

$$\vec{\nabla} \cdot \text{②} \quad : \quad \vec{\nabla} \cdot \frac{\partial}{\partial t} \vec{v}_1 = - \frac{v_s^2}{\Sigma_0} \nabla^2 \Sigma_{d1} - \nabla^2 \phi_1 + 2 \Omega \vec{\nabla} \vec{v}_1$$

we get

$$\boxed{\frac{\partial^2}{\partial t^2} \Sigma_{d1} - \underbrace{v_s^2 \nabla^2 \Sigma_{d1}}_{\text{pressure}} - \underbrace{\Sigma_0 \nabla^2 \phi_1}_{\text{gravity}} + \underbrace{2 \Sigma_0 \Omega \vec{\nabla} \vec{v}_1}_{\text{cent. force}} = 0}$$

Assume solutions of the form (waves) with $\vec{k} = k \vec{e}_x$

(no loss of generality isotropy)

$$\begin{cases} \Sigma_1(x,y,t) = \Sigma_a \, e^{i(\vec{k}\vec{x} - \omega t)} \\ \Sigma_{d1}(x,y,t) = \Sigma_{da} \, e^{i(\vec{k}\vec{x} - \omega t)} \\ \vec{v}_1(x,y,t) = (v_{ax} \vec{e}_x + v_{ay} \vec{e}_y) \, e^{i(\vec{k}\vec{x} - \omega t)} \\ \phi_1(x,y,z=0,t) = \phi_a \, e^{i(\vec{k}\vec{x} - \omega t)} \end{cases}$$

(! here, in the plane, $z=0$)

③ Poisson equation

$$\nabla^2 \phi_1 = 4\pi G \Sigma_1 \delta(z)$$

We want $\phi_1(x,y,z)$, such that

$$\begin{cases} \bullet\ z = 0 \quad : \quad \phi_1(x,y,z=0,t) = \phi_a \, e^{i(kx - \omega t)} \\ \bullet\ z \neq 0 \quad : \quad \nabla^2 \phi_1 = 0 \end{cases}$$

$$\Rightarrow \quad \phi_1(x,y,z,t) = \phi_a \, e^{i(kx - \omega t) - |kz|}$$

Solving Poisson links Σ_a with $\phi_a = - \dfrac{2\pi G \Sigma_a}{|k|}$

$$\Rightarrow \quad \underline{\phi_1(x,y,z,t) = - \frac{2\pi G \Sigma_a}{|k|} \, e^{i(kx - \omega t) - |kz|}}$$

Introducing $\Sigma_1(x,y,t)$, $\Sigma_{d1}(x,y,t)$, $\vec{v}_1(x,y,t)$, $\phi_1(x,y,z=0,t)$ in the linearized equations, we get the dispersion relation

$$\boxed{\omega^2 = v_s^2 k^2 - 2\pi G|k|\Sigma_0 \frac{\Sigma_a}{\Sigma_{da}} + 4\Omega^2 = 0}$$

Note that $\Sigma_a = \Sigma_{da} + \Sigma_e$

If we consider the evolution without the perturbation $\Sigma_e = 0$

$$\boxed{\omega^2 = v_s^2 k^2 - 2\pi G|k|\Sigma_0 + 4\Omega^2}$$

Interpretation

$$\boxed{\omega^2 = v_s^2 k^2 - 2\pi G|k|\Sigma_0 + 4\Omega^2}$$

$\omega^2 > 0$ STABLE

$\omega^2 < 0$ UNSTABLE

$\Rightarrow$ Ω helps to stabilize the slab

① $\boxed{\Omega^2 = 0}$ no rotation

$$\omega^2 = v_s^2 k^2 - 2\pi G |\vec{k}| \Sigma_0$$

$$= v_s^2 \left(k^2 - k_J |\vec{k}| \right)$$

⚠

$$k_J := \frac{2\pi G \Sigma_0}{v_s^2}$$

STABLE if $|\vec{k}| > k_J$

UNSTABLE if $|\vec{k}| < k_J$

homogeneous system

$$\omega^2 = v_s^2 k^2 - 4\pi G \rho_0$$

$$= v_s^2 \left(k^2 - k_J^2 \right)$$

⚠

$$k_J^2 := \frac{4\pi G \rho_0}{v_s^2}$$

② $\boxed{\Omega \neq 0 \quad , \quad v_s = 0}$ no pressure

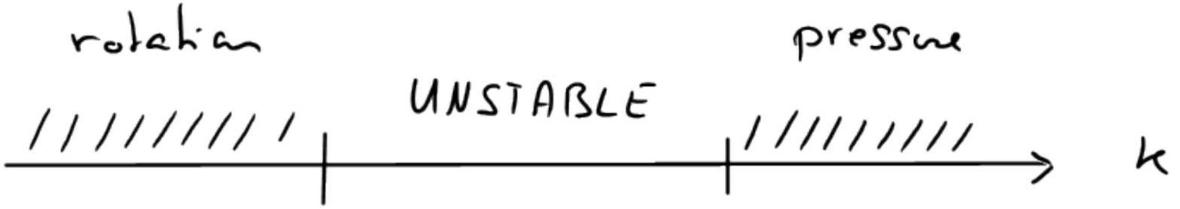
$$\omega^2 = 4\Omega^2 - 2\pi G |\vec{k}| \Sigma_0$$

STABLE if $|\vec{k}| < \dfrac{2\Omega^2}{\pi G \Sigma_0}$ (at large scale)

UNSTABLE if $|\vec{k}| > \dfrac{2\Omega^2}{\pi G \Sigma_0}$ (at small scale)

$\Rightarrow$ nothing can prevent the disk to be unstable at small scale

③ Complete stability

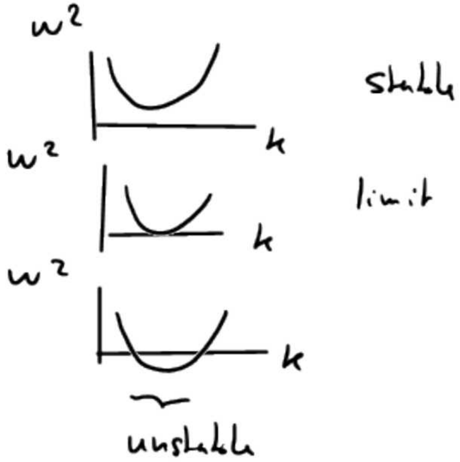


$$k_1 = \frac{2\Omega^2}{\pi G \Sigma_0} \qquad k_2 = \frac{2\pi G \Sigma_0}{V_s^2}$$

if $k_1 \geq k_2 \Rightarrow$ complete stability ! $\quad \frac{\Omega V_s}{G \Sigma_0} \gtrsim \pi$

More precisely

$$\omega^2 = V_s^2 k^2 - 2\pi G |\vec{k}| \Sigma_0 + 4\Omega^2 = 0$$



$$\boxed{\frac{V_s \Omega}{G \Sigma_0} \gtrsim \frac{1}{2}\pi} \simeq 1.57$$

The uniformly rotating stellar sheet

- if the velocity DF is Maxwellian

$$f_0\Big(\underset{\substack{\uparrow\\ \text{in the}\\ \text{plane}}}{\tilde{v}}\Big) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}}\, e^{-\frac{v^2}{2\sigma^2}}$$

- The stability critterion becomes

hydro

$$\boxed{\frac{\sigma\Omega}{G\Sigma_0} \gtrsim 1.68} \qquad\qquad \boxed{\frac{v_s\Omega}{G\Sigma_0} \gtrsim \frac{1}{2}\pi} \simeq 1.57$$

The stability of rotating disks :

the dispersion relation

(additional material)

The dispersion relation for a razor thin fluid disk

polar coordinates

$$\Sigma_d(R,\phi)\,,\quad \phi_d(R,\phi)\,,\quad p(R,\phi)\,,\quad \vec{v}(R,\phi) = v_R(R,\phi)\,\vec{e}_R + v_\phi(R,\phi)\,\vec{e}_\phi$$

① Continuity equation

$$\frac{\partial \Sigma_d}{\partial t} + \vec{\nabla}\left(\Sigma_d \vec{v}\right) = 0$$

$$\boxed{\frac{\partial \Sigma_d}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\,\Sigma_d v_R\right) + \frac{1}{R}\frac{\partial}{\partial \phi}\left(\Sigma_d v_\phi\right)}$$

② Euler equation

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\frac{\vec{\nabla} p}{\Sigma_d} - \vec{\nabla}\phi$$

$$\boxed{\begin{aligned} \frac{\partial v_R}{\partial t} + v_R\frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} &= -\frac{\partial \phi}{\partial R} - \frac{1}{\Sigma_d}\frac{\partial p}{\partial R} \\ \frac{\partial v_\phi}{\partial t} + v_R\frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_R}{R} &= -\frac{1}{R}\frac{\partial \phi}{\partial \phi} - \frac{1}{\Sigma_d}\frac{1}{R}\frac{\partial p}{\partial \phi} \end{aligned}}$$

③ Poisson $\nabla^2 \phi = 4\pi G \Sigma \delta(z)$

④ Equation of state (polytropic)

$p = K \Sigma_d^{\gamma}$ $\qquad v_s^2 = \gamma K \Sigma_0^{\gamma-1}$ (unperturbed sound speed)

$h = \frac{\gamma}{\gamma-1} K \Sigma_d^{\gamma-1}$ $\qquad \frac{\partial h}{\partial R} = \gamma K \Sigma_d^{\gamma-2} \frac{\partial \Sigma}{\partial R}$

(Specific enthalpy) $\qquad \frac{\partial h}{\partial \phi} = \gamma k \Sigma_d^{\gamma-2} \frac{\partial \Sigma}{\partial \phi}$

<u>The Euler Equation becomes</u>

$$\boxed{\left\{\begin{aligned} \frac{\partial V_R}{\partial t} + V_R \frac{\partial V_R}{\partial R} + \frac{V_\phi}{R}\frac{\partial V_R}{\partial \phi} - \frac{V_\phi^2}{R} &= -\frac{\partial}{\partial R}(\phi + h) \\ \frac{\partial V_\phi}{\partial t} + V_R \frac{\partial V_\phi}{\partial R} + \frac{V_\phi}{R}\frac{\partial V_\phi}{\partial \phi} + \frac{V_\phi V_R}{R} &= -\frac{1}{R}\frac{\partial}{\partial \phi}(\phi + h) \end{aligned}\right.}$$

Isolated system at equilibrium

Σ_{d0} , ϕ_0 , p_0 , $\vec{v}_0$ + axisymmetric

\+ $v_r = 0$ $\left(\frac{\partial}{\partial\phi} = 0\right)$

① Continuity equation $\quad 0 = 0$

② Euler equation $\quad \phi : \quad 0 = 0$

$$R : \quad \boxed{\underbrace{\frac{v_{\phi_0}^2}{R}}_{\text{centripetal acceleration}} = \frac{\partial}{\partial R}\left(\phi_0 + h_0\right) = \underbrace{\frac{\partial \phi_0}{\partial R}}_{\text{gravity force}} + \overbrace{\underbrace{v_s^2 \frac{d}{dR} \ln \Sigma_0}_{\text{pressure force}}}^{\circledast}}$$

Notes

① $\circledast$ $\quad v_s^2 \frac{d}{dR}\ln\Sigma_0 = \left.\frac{\partial p}{\partial \Sigma}\right|_{\Sigma_0} \frac{1}{\Sigma_0}\frac{\partial \Sigma_0}{\partial R} = \frac{1}{\Sigma_0}\frac{\partial p}{\partial R} = \frac{1}{\Sigma_0}\vec{\nabla} p$

② $\left.\begin{array}{l} v_s \simeq 10 \text{ km/s} \\ v_\phi \simeq 100 \text{ km/s} \end{array}\right\} \Rightarrow$ we neglect the pressure

③ $v_{\phi_0} \simeq \sqrt{R \frac{\partial \phi_0}{\partial R}} = R\,\Omega(R)$

The response of the system to a weak perturbation $\quad -\varepsilon \vec{\nabla} \phi_e$

$$\left.\begin{aligned}
\Sigma_{d0} &\rightarrow \Sigma_{d0} + \varepsilon \Sigma_{d1} \\
v_{R_0} = 0 &\rightarrow \varepsilon v_{R1} \\
v_{\phi 0} &\rightarrow v_{\phi_0} + \varepsilon v_{\phi_1} \\
h_0 &\rightarrow h_0 + \varepsilon h_1 \\
\phi_{d0} &\rightarrow \phi_{d0} + \varepsilon \phi_{d1}
\end{aligned}\right\}$$

disk only
without the
perturbation

$$\overset{\phi_{d0}}{\parallel}$$

$$\left.\phi_0 \rightarrow \phi_{d0} + \underbrace{\varepsilon \phi_1}_{\varepsilon \phi_{d1} + \varepsilon \phi_e}\right\}$$

total potential,
includes the perturbation

Linearized equations for a razor thin fluid disk

① Continuity equation

$$\frac{\partial}{\partial t}\Sigma_{d1} + \Omega\frac{\partial \Sigma_{d1}}{\partial \phi} + \frac{1}{R}\frac{\partial}{\partial R}\left(R v_{R1}\Sigma_0\right) + \frac{\Sigma_0}{R}\frac{\partial v_{\phi 1}}{\partial \phi} = 0$$

② Euler equation

$$\frac{\partial v_{R1}}{\partial t} + \Omega\frac{\partial v_{R1}}{\partial \phi} - 2\Omega v_{\phi 1} = -\frac{\partial}{\partial R}\left(\Phi_1 + h_1\right)$$

$$\frac{\partial v_{\phi 1}}{\partial t} + \underbrace{\left[\frac{d}{\partial R}\left(\Omega R\right) + \Omega\right]}_{-2B(R)} v_{R1} + \Omega\frac{\partial v_{\phi 1}}{\partial \phi} = -\frac{1}{R}\frac{\partial}{\partial \phi}\left(\phi_1 + h_1\right)$$

Oort constant $\qquad \kappa^2 = -4B\Omega$

Solutions

assume waves of the form (can be a spiral)

5 radial functions

$$\underline{\Sigma_{d1}} = Re\left[\Sigma_{da}(R)\, e^{i(m\phi-\omega t)} \right]$$

$$\underline{v_{R1}} = Re\left[v_{Ra}(R)\, e^{i(m\phi-\omega t)} \right]$$

$$\underline{v_{\phi 1}} = Re\left[v_{\phi a}(R)\, e^{i(m\phi-\omega t)} \right]$$

$$\underline{h_1} = Re\left[h_a(R)\, e^{i(m\phi-\omega t)} \right]$$

without perturbation

$$\Sigma_1 = Re\left[\Sigma_a(R)\, e^{i(m\phi-\omega t} \right]$$

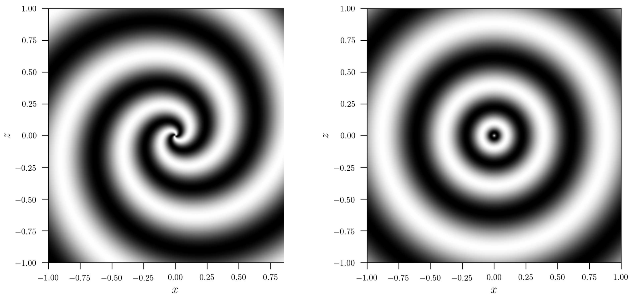
$$\underline{\phi_1} = Re\left[\phi_a(R)\, e^{i(m\phi-\omega t} \right]$$

total: disk + perturbation

$$\Sigma_a = \Sigma_{da} + \Sigma_e$$

$$\phi_a = \phi_{da} + \phi_e$$

$$\Sigma_{d1}(R,\phi,t) = Re\left[H_{da}(R,t)\, e^{i(m\phi+f(R,t)-\omega t)} \right]$$



① The continuity equation gives

$$\boxed{-i(\omega - m\Omega)\underline{\Sigma_{da}} + \frac{1}{R}\frac{d}{dR}\left(R\,\underline{v_{Ra}}\,\Sigma_0\right) + \frac{im\Sigma_0}{R}\underline{v_{\phi a}} = 0}$$

② $\underline{\text{The Euler equation gives}}$

$$\boxed{\begin{cases} \underline{v_{Ra}}(R) = \frac{i}{\Delta}\left[(\omega - m\Omega)\frac{d}{dR}(\underline{\phi_a} + \underline{h_a}) - \frac{2m\Omega}{R}(\underline{\phi_a} + \underline{h_a}) \right] \\ \underline{v_{\phi a}}(R) = -\frac{i}{\Delta}\left[2B\frac{d}{dR}(\underline{\phi_a} + \underline{h_a}) + \frac{m(\omega - m\Omega)}{R}(\underline{\phi_a} + \underline{h_a}) \right] \end{cases}}$$

with $\underline{\Delta = \kappa^2 - (\omega - m\Omega)^2}$

$\underline{\text{Notes}}$ ① v_{Ra}, $v_{\phi a}$, κ, Δ, Ω are fonctions of R only

② if $\Delta = 0 \Rightarrow$ divergeance

$\Omega_p = \frac{\omega}{m}$ pattern speed

$$\kappa^2 = (\omega - m\Omega)^2$$

$$= m^2\left(\frac{\omega}{m} - \Omega\right)^2 = m^2(\Omega_p - \Omega)^2$$

$$\underline{\Omega_p = \Omega \pm \frac{\kappa}{m}}$$

!!! problems at the Lindblad resonance

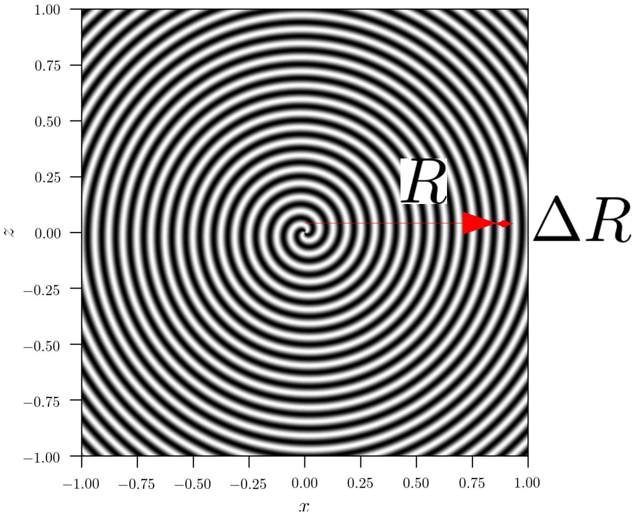
④ "Equation of state"

$h = \frac{\gamma}{\gamma-1} K \Sigma_d^{\gamma-1}$ $v_s^2 = \gamma K \Sigma_0^{\gamma-2}$

$$\boxed{\underline{h_a} = \gamma K \Sigma_0^{\gamma-2} \underline{\Sigma_{da}} = v_s^2 \frac{\underline{\Sigma_{da}}}{\Sigma_0}}$$

=> we have 4 equations for 5 unknowns $\Sigma_{da}, h_a, \phi_a, v_{Ra}, v_{\phi a}$

⑤ Poisson Equation

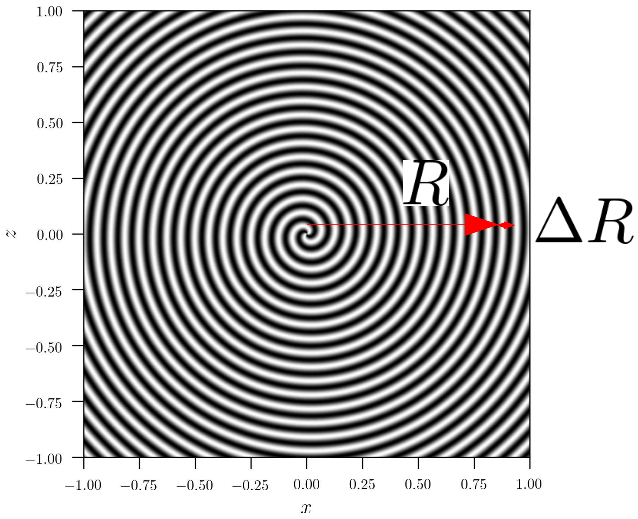


Stability of collisionless systems

The WKB approximation

Potential of tightly wound spiral pattern

Tightly wound spiral approximation



We assume that $\underline{\Delta R \ll R}$

but $\Delta R = \frac{2\pi}{m|k|} \sim \frac{1}{|k|}$

Thus : $\boxed{|kR| \gg 1}$

WKB approximation

(Wentzel - Kramers - Brillouin)

Model of a spiral surface density

$$\Sigma_d(R,\phi,t) = \underbrace{\Sigma_{d0}(R)}_{\text{at equilibrium}} + \varepsilon\, \underbrace{\Sigma_{d1}(R,\phi,t)}_{\substack{\text{non axi-symetric perturbation} \\ \text{(spiral pattern)}}}$$

Lets formulate Σ_1 as

$$\Sigma_1(R,\phi,t) = R_e\left(\underbrace{H(R,t)}_{\substack{\text{"slow" radial} \\ \text{variation} \\ \text{amplitude of} \\ \text{the spiral arm}}} \underbrace{e^{i(m\phi + \overset{\text{Shape function}}{f}(R,t))}}_{\text{"rapid" variation}} \right)$$

What is the corresponding potential ?

"rapid" radial variation of $\Sigma(R)$

$\Rightarrow$ the contribution of the distant parts cancels

$$\boxed{\Rightarrow \text{ the potential is determined from } \Sigma_1(R,\phi,t) \text{ within a few } k}$$

<u>Method</u> : expand the shape function around $(\underbrace{R_0, \phi_0})_{\text{any "random point"}}$

$$\left\{ \begin{aligned} \bullet \quad f(R,t) &\cong f(R_0,t) + \frac{\partial f}{\partial R}(R-R_0) \\ &= f(R_0,t) + k(R_0,t)(R-R_0) \qquad k = |\vec{k}| \\ \bullet \quad H(R,t) &\cong H(R_0,t) \end{aligned} \right.$$

We get :

$k = k(R_0,t)$

$$\Sigma_1(R,\phi,t) \simeq H(R_0,t)\, e^{i(m\phi_0 + f(R_0,t) + k(R-R_0))}$$

$$= H(R_0,t)\, e^{i(m\phi_0 + f(R_0,t))}\, e^{ik(R-R_0)}$$

$$= \underbrace{\Sigma_a}_{\text{no radial dependency}}\, e^{ik(R-R_0)}$$

this is a plane wave with

$\vec{k} = k\,\vec{e}_R$

(see the rotating sheet)

The corresponding potential is thus

$$\phi_1(R,\phi,z,t) \simeq -\frac{2\pi G \Sigma_a}{k}\, e^{ik(R-R_0) - |kz|}$$

$$\simeq -\frac{2\pi G}{k}\, \underbrace{H(R_0,t)\, e^{i(m\phi_0 + f(R_0,t))}}_{\Sigma_a}\, e^{ik(R-R_0) - |kz|}$$

We are free to set $R = R_0$, $\phi = \phi_0$, $z = 0$ and thus, we get

$$\boxed{\Phi_n(R, \phi, z, t) = - \frac{2\pi G}{k} H(R,t) \, e^{i[m\phi + f(R,t)]}}$$

Stability of collisionless systems

Back to the dispersion relation

④ "Equation of state" $h = \frac{\gamma}{\gamma-1} K \Sigma_d^{\gamma-1}$ $v_s^2 = \gamma K \Sigma_0^{\gamma-1}$

$$\boxed{\underline{h_a} = \gamma K \Sigma_0^{\gamma-2} \underline{\Sigma_{da}} = v_s^2 \frac{\underline{\Sigma_{da}}}{\Sigma_0}}$$

=> we have 4 equations for 5 unknowns $\Sigma_{da}, h_a, \phi_a, v_{Ra}, v_{\phi a}$

⑤ Poisson Equation use WKB approximation

$$\begin{aligned}
\Sigma_1 &= R_e\left[\Sigma_a(R)\, e^{i(m\phi-\omega t)}\right] \\
&= R_e\left[e^{-i\omega t}\, \Sigma_a(R)\, e^{im\phi}\right] \\
&= R_e\left[H(R,t)\, e^{if(R,t)}\, e^{im\phi}\right]
\end{aligned}$$

$$\boxed{\begin{gathered} k = \frac{\partial f}{\partial R} \\ |kR| \gg 1 \end{gathered}}$$

$$\phi_1 = -\frac{2\pi G}{k}\, \underbrace{H(R,t)}_{\text{slow variation}}\, \underbrace{e^{if(R,t)}}_{\text{fast variation}}\, \underbrace{e^{im\phi}}_{\text{azimuthal symmetry}}$$

In addition

④ "Eq. of state" $\Rightarrow$ $h_a \sim \Sigma_{da} \sim \phi_1 \sim e^{if(R,t)}$

Thus a) $\frac{d}{dR}(\phi_a + h_a) \sim (\phi_a + h_a)\, i \frac{d}{dR} f = (\phi_a + h_a)\, ik$

b) $\frac{1}{R}(\phi_a + h_a) \cong \frac{1}{R}(\phi_a + h_a)\, ik \frac{1}{ik} = \frac{-i}{Rk} \frac{d}{dR}(\phi_a + h_a)$

as $Rk \gg 1$ $\quad \underline{\frac{1}{R}(\phi_a + h_a) \ll \frac{d}{dR}(\phi_a + h_a)}$

① The continuity equation becomes $\quad \frac{d}{dR}(R \Sigma_0 v_{Ra}) = ik R \Sigma_0 v_{Ra}$

$$\boxed{-(\omega - m\Omega)\, \underline{\Sigma_{da}} + k \Sigma_0 \underline{v_{Ra}} = 0}$$

② The Euler equation becomes $\quad$ + EoS $(h_a \sim \Sigma_{da})$ + poisson $(\Sigma_a \sim \phi_a)$

$$\boxed{\begin{aligned} \underline{v_{Ra}} &= -\frac{\omega - m\Omega}{\Delta} k (\underline{\phi_a} + \underline{h_a}) \\ \underline{v_{\phi a}} &= -\frac{2iB}{\Delta} k (\underline{\phi_a} + \underline{h_a}) \end{aligned}}$$

We can solve to get

$$\boxed{\Sigma_{da} = \left(\frac{2\pi G \Sigma_0 |k|}{\kappa^2 - (\omega - m\Omega)^2 + v_s^2 k^2} \right) \Sigma_a}$$

Which gives the dispersion relation if $\Sigma_e = 0$ $\left(\Sigma_{da} = \Sigma_a \right)$

$$\boxed{(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2}$$

<u>Note:</u> if $\Omega(R) = \text{cte}$ $\kappa = 2\Omega$ and we recover the dispersion relation for uniformly rotating sheet

$$\boxed{\omega^2 = 4\Omega^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2}$$

Interpretation

$$\boxed{(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2}$$

Axisymmetric perturbations

$m = 0$

"Cold disk"

$v_s = 0$

$$\boxed{\omega^2 = \kappa^2 - 2\pi G \Sigma_0 |k|}$$

$$k_{crit} := \frac{\kappa^2}{2\pi G \Sigma_0} \qquad \lambda_{crit} := \frac{2\pi}{k_{crit}} = \frac{4\pi^2 G \Sigma_0}{\kappa^2}$$

$|k| > k_{crit}$ $\quad (\lambda < \lambda_{crit})$ $\quad \omega^2 < 0$ $\quad$ **UNSTABLE**

$|k| < k_{crit}$ $\quad (\lambda > \lambda_{crit})$ $\quad \omega^2 > 0$ $\quad$ **STABLE**

rotation stabilizes
at large scale

$$\boxed{(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2}$$

"Non rotating fluid disk"

$\kappa = 0$

$$\boxed{\omega^2 = -2\pi G \Sigma_0 |k| + v_s^2 k^2}$$

⚠ not realistic

$$k_{crit} := \frac{2\pi G \Sigma_0}{v_s^2} \quad (= k_{Jeans}) \qquad \lambda_{crit} = \frac{2\pi}{k_{crit}} = \frac{v_s^2}{G \Sigma_0}$$

$|k| > k_{crit}$ $(\lambda < \lambda_{crit})$ $\omega^2 > 0$ **STABLE**

$|k| < k_{crit}$ $(\lambda > \lambda_{crit})$ $\omega^2 < 0$ **UNSTABLE**

pressure
stabilizes the disk

$$\boxed{(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2}$$

Rotating fluid disk

$$\boxed{\omega^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2}$$

$$k_1 = \frac{2\kappa^2}{\pi G \Sigma_0} \qquad k_2 = \frac{2\pi G \Sigma_0}{v_s^2}$$

Complete stability

$$k_1 \gg k_2$$

$$\boxed{Q := \frac{\kappa v_s}{\pi G \Sigma_0} > 1}$$

Stability of stellar disks

$$\boxed{Q := \frac{\varkappa v_s}{\pi G \Sigma_0} > 1}$$

$$\boxed{Q := \frac{\varkappa \sigma_R}{3.36 \, G \Sigma_0} > 1}$$

Safronov - Toomre criterion

Non axisymmetric perturbations ①

$$\boxed{(\omega - m\Omega)^2 = \varkappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2}$$

$$\omega = \sqrt{\varkappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2} + m\Omega$$

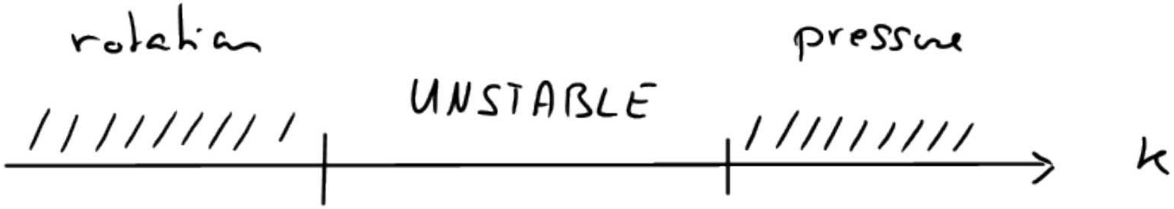
- the stability is determined for $m = 0$ (value of $\sqrt{\ }$)
- $m\Omega \in \mathbb{R}_e$ → add an oscillatory term $e^{-i m\Omega}$

  with a frequency that correspond to

  the passage of spiral arm

  (ex : if $\Gamma = 0$)

Non axisymmetric perturbations ②



$$k_1 = \frac{\kappa^2}{2\pi G \Sigma_0} \qquad k_2 = \frac{2\pi G \Sigma_0}{v_s^2}$$

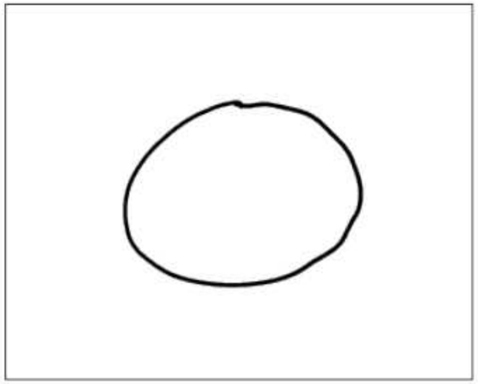
- $\lambda_1 = \frac{2\pi}{k_1} = 4\pi^2 \frac{G\Sigma_0}{\kappa^2}$ = "max size" of a "clump"
- number of "clumps" $\frac{2\pi R}{\lambda_1} = R \cdot k = \frac{\kappa^2 R}{2\pi G \Sigma_0} = m$

→ number of spiral arms

$$\boxed{X = \frac{\kappa^2 R}{2\pi G \Sigma m} \geqslant 3}$$

↑
ensure the stability of mode m

Physical interpretation of the Jeans instability



Specific potential energy of an homogeneous sphere of radius r

$$|W| \cong \frac{G M(r)}{r} \qquad M(r) = \frac{4}{3}\pi r^3 \rho_0$$

$$\cong \frac{4}{3}\pi r^2 \rho_0 G$$

$r > r_J$ $|W| > 2T$

UNSTABLE

$r < r_J$ $|W| < 2T$

STABLE

Specific kinetic energy of an homogeneous sphere of radius r

$$K \cong \frac{1}{2}\sigma^2$$

Radius for which we have the virial equilibrium $2T = |W|$

$$r = \sqrt{\frac{3\sigma^2}{4G\rho_0\pi}} \sim \lambda_J = \sqrt{\frac{\pi v_s^2}{G\rho_0}}$$