The Jaynes - Cummings Hamiltonian.

This is a theoretical model of a two level system coupled to an harmonic oscillator. It has many applications. For example:

- Atom in a Cavity interacting with a mode of the electromagnetic field trapped in the cavity. Atom is approx by two level system and e-m mode is the harm osc.

- qubits based on LC circuits (with superconducting Josephson junction for L) interacting with e-m mode of resonators (wave guide) coupled to the circuit. Important physics behind superconducting Qubit platforms (in NISQ devices).

Here we give a brief and simple introduction to the Hamiltonian itself. More advanced theory forms the basis to model interesting quantum effects: study of superposition states, their decoherence, entanglement, probability revivals, ...

We adopt the language of the $\underbrace{\text{"atom"}}_{\text{two level system}}$ - $\underbrace{\text{"field"}}_{\text{harm osc mode}}$.

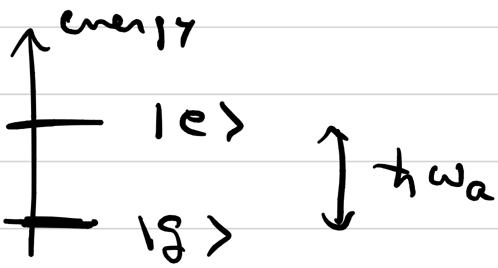
Hamiltonian has three terms

$$\hat{H} = H_{\substack{\text{two level}\\ \text{atom}}} + H_{\substack{\text{harm osc}\\ \text{or field}}} + H_{\text{interaction}}$$

$$= \frac{\hbar \omega_a}{2} \sigma_z + \hbar \omega \left(a^\dagger a + \frac{1}{2}\right) + H_{\text{interaction}}$$

<u>Two level system</u> ; states $|g\rangle$ and $|e\rangle$

"(atom)" "ground state" "excited states"



$$H_{\text{two level}} = \frac{\hbar\omega_a}{2}\sigma_z = \frac{\hbar\omega_a}{2}\left(|e\rangle\langle e| - |g\rangle\langle g|\right)$$

$$\sigma_z = \begin{array}{c c} & \begin{array}{cc} e & g \end{array} \\ \begin{array}{c} e \\ g \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array} = |e\rangle\langle e| - |g\rangle\langle g|$$

<u>Harmonic osc (field mode)</u>

$$H_{\text{harm osc}} = \hbar\omega\left(a^{\dagger}a + \frac{1}{2}\right) \rightsquigarrow \hbar\omega\, a^{\dagger}a$$

$$= \hbar\omega \sum_{n=0}^{+\infty}\left(n+\frac{1}{2}\right)|n\rangle\langle n| .$$

Recall $a^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$ creation op $a^{\dagger}$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$ annihilation op a.

$a^{\dagger}a|n\rangle = n|n\rangle$ $a^{\dagger}a =$ number op

Interaction term

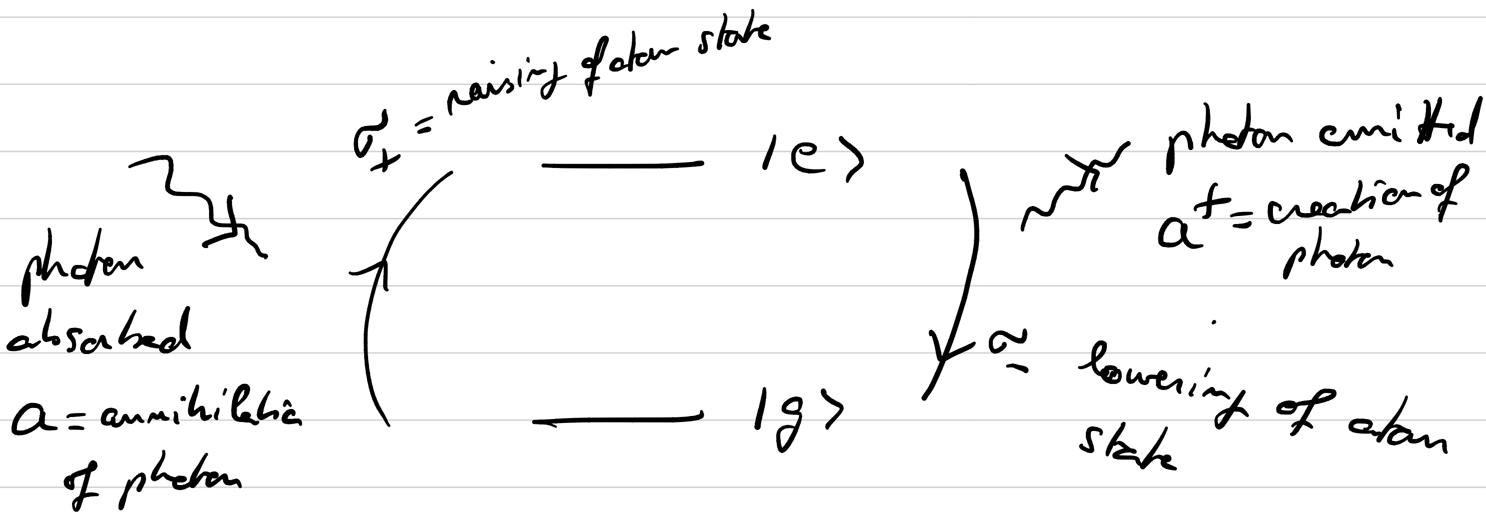
$$H_{int} = \frac{\hbar\Omega}{2}\left(a^{+}\sigma_{-} + a\sigma_{+}\right)$$

where

$$\sigma_{-} = \sigma_x - i\sigma_y = \begin{array}{c|cc} & e & g \\ \hline e & 0 & 0 \\ g & 1 & 0 \end{array} = |g\rangle\langle e|$$

$$\sigma_{+} = \sigma_x + i\sigma_y = \begin{array}{c|cc} & e & g \\ \hline e & 0 & 1 \\ g & 0 & 0 \end{array} = |e\rangle\langle g|$$

main idea behind this coupling term:



for example:

$$\begin{cases} a^{+}\sigma_{-}\,|e, n\rangle = \sqrt{n+1}\,|g, n+1\rangle \\ a\sigma_{+}\,|g, n\rangle = \sqrt{n}\,|e, n-1\rangle \end{cases}$$

(Note: full treatment → other terms which are neglected here).

Total Hamiltonian (simplest possible form of Jaynes - Cumming hamiltonian)

$$\hat{H} = \frac{\hbar\omega_a}{2}\sigma_z + \hbar\omega\, a^{\dagger}a + \frac{\hbar\Omega}{2}\left(\sigma_+ a + \sigma_- a^{\dagger}\right).$$

(we drop the constant factor $\frac{\hbar\omega}{2}$ in $\hbar\omega(a^{\dagger}a + \frac{1}{2})$).

Conserved Quantity: $a^{\dagger}a + \sigma_+\sigma_-$.

$$\sigma_+\sigma_- = |e\rangle\langle g|\,|g\rangle\langle e| = |e\rangle\langle e| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

counts 1 if atom is excited and 0 if not excited

$a^{\dagger}a$ = number of exc in harm osc (nb of photons in field).

$\Rightarrow$ $a^{\dagger}a + \sigma_+\sigma_-$ = number of "excitations" of atom + field system.

Note: $(a^{\dagger}a + \sigma_+\sigma_-)|e, n\rangle = (n+1)|e, n\rangle$

$(a^{\dagger}a + \sigma_+\sigma_-)|g, n+1\rangle = (n+1)|g, n+1\rangle.$

So we see that $(a^+a + \sigma_+\sigma_-)$ leaves invariant the subspace $\alpha\,|e,n\rangle + \beta\,|g,n+1\rangle$.

$$\underbrace{\text{subspace}}$$

subspace with $n+1$ "excitations".

How does H act on this subspace?

$$\begin{cases} H\,|e,n\rangle = \frac{\hbar\omega_a}{2}|e,n\rangle + \hbar\omega\sqrt{n}\,|e,n\rangle + \overbrace{\frac{\hbar\Omega_0}{2}}^{\sqrt{n+1}}|g,n+1\rangle \\ H\,|g,n+1\rangle = -\frac{\hbar\omega_a}{2}|g,n+1\rangle + \hbar\omega\sqrt{n+1}\,|g,n+1\rangle + \frac{\hbar\Omega_0}{2}\sqrt{n+1}\,|e,n\rangle \end{cases}$$

So $H\big(\alpha\,|e,n\rangle + \beta\,|g,(n+1)\rangle\big) = \gamma\,|e,n\rangle + \delta\,|g,n+1\rangle$

H also leaves invariant this same subspace.

Remark: in fact $[H, a^+a + \sigma_+\sigma_-] = 0$ which means that H and $a^+a + \sigma_+\sigma_-$ have the same invariant subspace & can be diagonalized in same basis.

Diagonalisation of H.

The preceding remarks show that H is block diagonal of the form:

$$
\begin{array}{c|ccccccc}
 & |g,0\rangle & |g,1\rangle & |e,0\rangle & |g,2\rangle & |e,1\rangle & \cdots \\
\hline
|g,0\rangle & \boxed{\times} & & & & & \\
|g,1\rangle & & \times & \times & & & \\
|e,0\rangle & & \times & \times & & & \\
|g,2\rangle & & & & \times & \times & \\
|e,1\rangle & & & & \times & \times & \\
\vdots & & & & & & \ddots
\end{array}
$$

Top element :

$$\langle g,0|\,H\,|g,0\rangle = -\frac{\hbar\omega_a}{2} + 0 + 0 = -\frac{\hbar\omega_a}{2}.$$

Blocks $n = 0, 1, 2, 3 \dots$

$$\begin{bmatrix} \langle g\,n{+}1|H|g\,n{+}1\rangle & \langle g\,n{+}1|H|e\,n\rangle \\ \langle e\,n|H|g\,n{+}1\rangle & \langle e\,n|H|e\,n\rangle \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\hbar\omega_a}{2} + \hbar\omega(n+1) & \frac{\hbar\Omega}{2}\sqrt{n+1} \\ \frac{\hbar\Omega}{2}\sqrt{n+1} & +\frac{\hbar\omega_a}{2} + \hbar\omega n \end{bmatrix}$$

↑

check exercise!

Eigenvalues are: $m \geq 0$

$$E_\pm(m) = \hbar\omega\left(m+\frac{1}{2}\right) \pm \frac{1}{2}\hbar\sqrt{\delta^2 + \Omega^2(m+1)}$$

$\delta = \omega_a - \omega$ (detuning parameter between atom & field frequency.).

Eigenvectors or eigenstates are: $m \geq 0$.

$$\begin{cases} |+, m\rangle = \cos\frac{\theta_m}{2}|e, m\rangle + \sin\frac{\theta_m}{2}|g, m+1\rangle \\ |-, m\rangle = \sin\frac{\theta_m}{2}|e, m\rangle - \cos\frac{\theta_m}{2}|g, m+1\rangle \end{cases}$$

with $\tan\theta_m = \dfrac{\Omega\sqrt{m+1}}{\delta}$.

(and for $|g, 0\rangle$ energy is $-\frac{\hbar\omega_a}{2}$).

Suppose we inject an atom in initial state

$|e\rangle$ in some state $\sum_n C_n\,|n\rangle$ of e-m field

$$|\psi_0\rangle = \sum_n C_n\,|e,n\rangle .$$

$$|\psi(t)\rangle = e^{-i\frac{t}{\hbar}\hat{H}}|\psi_0\rangle = \sum_n C_n\, e^{-i\frac{t}{\hbar}\hat{H}}|e,n\rangle$$

↑

state of atom + field

at time t

since $|e,n\rangle = \cos\frac{\theta_n}{2}\,|+,n\rangle + \sin\frac{\theta_n}{2}\,|-,n\rangle$

$$e^{-i\frac{t}{\hbar}\hat{H}}|e,n\rangle = \cos\frac{\theta_n}{2}\, e^{-i\frac{t}{\hbar}E_+(n)}|+,n\rangle + \sin\frac{\theta_n}{2}\, e^{-i\frac{t}{\hbar}E_-(n)}|-,n\rangle$$

$\Rightarrow$

$$|\psi(t)\rangle = \sum_n C_n\left[\left(\cos\frac{\theta_n}{2}\right) e^{-i\frac{t}{\hbar}E_+(n)}|+,n\rangle + \sin\theta_n\, e^{+i\frac{t}{\hbar}E_-(n)}|-,n\rangle\right]$$

From this formula we can study

Rabi oscillations of system "atom+field".

For example let us look at so-called Rabi

oscillations in vacuum state:

- e-m field is initially in $\sum c_n |n\rangle = |0\rangle$

(No photons = vacuum initially).

- atom initially in $|e\rangle$ = excited state,

- Initial state is $|e, 0\rangle$.

Let us look at tuned situation $\delta = 0 \Leftrightarrow \omega_0 = \omega$

we have $$E_{\pm}(0) = \frac{\hbar\omega}{2} \pm \frac{\hbar\Omega}{2}.$$

$$\tan\vartheta_0 = \frac{\hbar\Omega}{\delta} \to +\infty \Rightarrow \vartheta_0 \to \frac{\pi}{2}.$$

$$|e, 0\rangle = \frac{1}{\sqrt{2}}\left(|+, 0\rangle + |-, 1\rangle\right)$$

$$\Rightarrow |\psi(t)\rangle = e^{-\frac{i\omega t}{2}} \left\{ e^{i\frac{\hbar\Omega t}{2}} |+,0\rangle + e^{-i\frac{\hbar\Omega t}{2}} |-,0\rangle \right\}$$

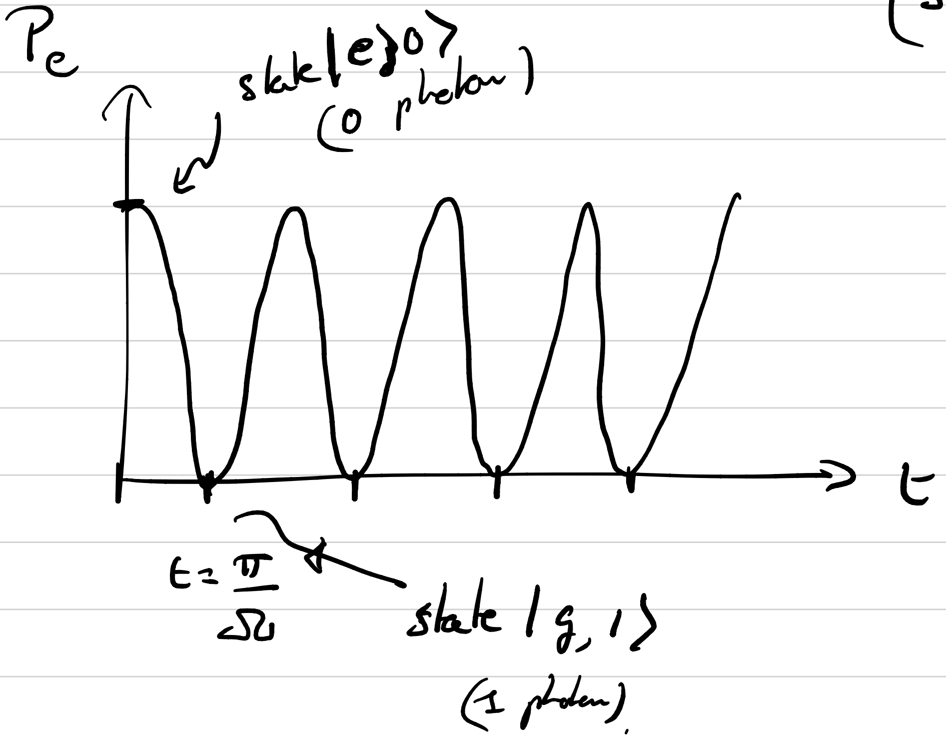
$$\propto \cos\left(\frac{\Omega t}{2}\right) |e,0\rangle - i \sin\left(\frac{\Omega t}{2}\right) |g,1\rangle$$

P_e = Prob (to see state $|e\rangle$ at time t) $= |\langle e,0 | \psi(t)\rangle|^2$

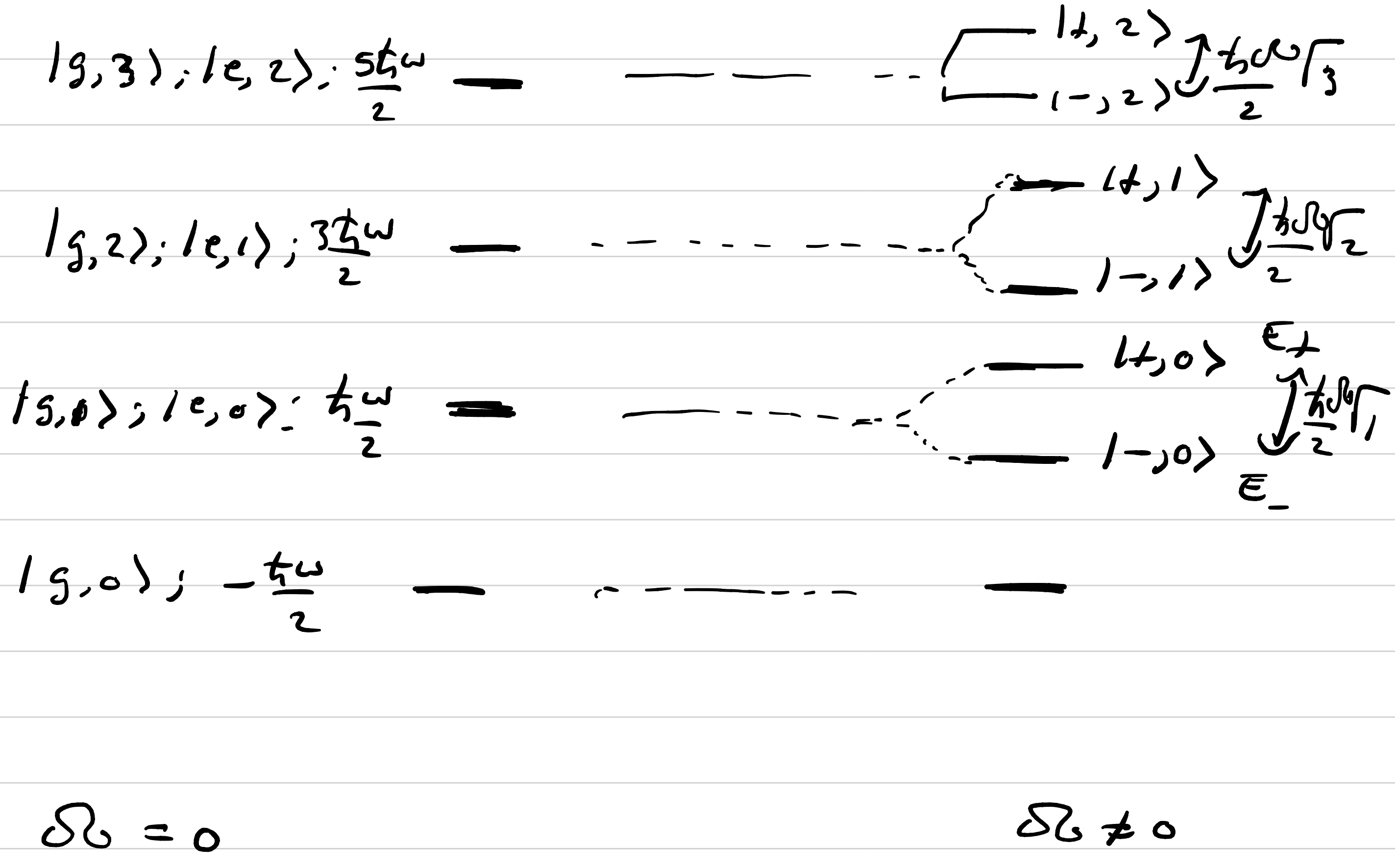
$$= \left(\cos \frac{\Omega t}{2}\right)^2$$

P_g = Prob (to see state $|g\rangle$ at time t) $= \underline{|\langle g,1 | \psi(t)\rangle|^2}$

$$= \left(\sin \frac{\Omega t}{2}\right)^2.$$



Jaynes - Cummings ladder. $\boxed{\omega = \omega_a}$ tuned situation.



$$\hat{H} = \frac{\hbar\omega_a}{2}\sigma_z + \hbar\omega a^\dagger a$$

$$\begin{cases} \omega_a = \omega \\ \delta = 0 \end{cases}$$

add

$$H_{int} = \frac{\hbar\Omega}{2}\left(a^\dagger\sigma_- + a\sigma_+\right)$$

lift degeneracy with n

gaps $\sim \sqrt{n}\,\frac{\hbar\Omega}{2}$.

(Non linear growth)