Solutions Series 12 - Exterior differentials, Orientations, Integration on Manifolds **2023–01–03**

Exercise 12.1. Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the smooth map

$$F(\theta, \varphi) = ((\cos\varphi + 2)\,\cos\theta, (\cos\varphi + 2)\,\sin\theta, \sin\varphi).$$

Consider $\omega = y\,\mathrm{d}z \wedge \mathrm{d}x$. Compute $\mathrm{d}\omega$ and $F^*\omega$ and verify by direct computation that $\mathrm{d}(F^*\omega) = F^*\,\mathrm{d}\omega$.

Solution. We first note that $\mathrm{d}\omega = \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x$. To obtain $F^*\omega$ we compute first

$$F^*(\mathrm{d}z) = \mathrm{d}\sin\varphi = \cos\varphi\,\mathrm{d}\varphi$$

and then

$$\begin{aligned} F^*(\mathrm{d}x) &= \mathrm{d}((\cos\varphi + 2)\cos\theta) \\ &= -\sin\varphi\cos\theta\,\mathrm{d}\varphi - (\cos\varphi + 2)\sin\theta\,\mathrm{d}\theta \end{aligned}$$

Putting everything together, we get

$$\begin{aligned} F^*\omega &= (\cos\varphi + 2)\,\sin\theta\ \ \cos\varphi\,\mathrm{d}\varphi \wedge (-\sin\varphi\cos\theta\,\mathrm{d}\varphi - (\cos\varphi + 2)\sin\theta\,\mathrm{d}\theta) \\ &= -(\cos\varphi + 2)^2\,(\sin\theta)^2\,\mathrm{d}\varphi \wedge \mathrm{d}\theta. \end{aligned}$$

The forms $\mathrm{d}(F^*\omega)$ and $F^*(\mathrm{d}\omega)$ are both zero because they are 3-forms on $\mathbb{R}^2$. □

Exercise 12.2. Prove that given $\omega \in \Omega^k(\mathbb{R}^n)$ a k-form the Exterior derivative $\mathrm{d}\omega = \mathrm{d}(\sum_I \omega_I\,\mathrm{d}x^I) := \sum_I d\omega_I \wedge dx^I$ has the following properties:

(a) d is $\mathbb{R}$-linear

Solution. This one is easy. □

(b) $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^{\deg(\omega)}\omega \wedge \mathrm{d}\eta$

Solution. Let $a = \deg\omega$ and $b = \deg\eta$. The form ω can be expressed in standard form as $\omega = \sum_I \omega_I\ \mathrm{d}x^I$, where the sum ranges over all increasing multiindices $I = (i_0, \ldots, i_{a-1})$. Similarly, we have $\eta = \sum_J \eta_J\ \mathrm{d}x^J$ for increasing multiindices $J = (j_0, \ldots, j_{b-1})$. Hence we can write

$$\begin{aligned} \omega \wedge \eta &= \left(\sum_I \omega_I\ \mathrm{d}x^I\right) \wedge \left(\sum_J \eta_J\ \mathrm{d}x^J\right) \\ &= \sum_{I,J} \omega_I\,\eta_J\ \mathrm{d}x^I \wedge \mathrm{d}x^J \end{aligned}$$

and we note that $\mathrm{d}x^I \wedge \mathrm{d}x^J$ is equal to $\pm\,\mathrm{d}x^K$ (for some K) if the multiindices I and J have no common values, and vanishes otherwise. This means that the the form $\omega \wedge \eta$ is already expressed in standard form, so its exterior derivative can be computed as usual:

$$
\begin{aligned}
\mathrm{d}(\omega \wedge \eta) &= \mathrm{d}\left(\sum_{I,J} \omega_I\, \eta_J \;\mathrm{d}x^I \wedge \mathrm{d}x^J\right) \\
&= \sum_{I,J} \mathrm{d}(\omega_I\, \eta_J) \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J) \\
&= \sum_{I,J} (\eta_J \;\mathrm{d}\omega_I + \omega_I \;\mathrm{d}\eta_J) \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J \\
&= \sum_{I,J} \eta_J \,\mathrm{d}\omega_I \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J + \sum_{I,J} \omega_I \;\mathrm{d}\eta_J \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J \\
&= \sum_{I,J} \eta_J \,\mathrm{d}\omega_I \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J + (-1)^b \sum_{I,J} \omega_I \;\mathrm{d}x^I \wedge \mathrm{d}\eta_J \wedge \mathrm{d}x^J \\
&= \left(\sum_I \mathrm{d}\omega_I \wedge \mathrm{d}x^I\right) \wedge \left(\sum_J \eta_J \,\mathrm{d}x^J\right) + (-1)^b \left(\sum_I \omega_I \;\mathrm{d}x^I\right) \wedge \left(\sum_J \mathrm{d}\eta_J \wedge \mathrm{d}x^J\right) \\
&= \mathrm{d}\omega \wedge \eta + (-1)^b \omega \wedge \mathrm{d}\eta
\end{aligned}
$$

□

(c) $\mathrm{d} \circ \mathrm{d} = 0$

Solution. Take any a form ω and write $\omega = \sum_I \omega_I \;\mathrm{d}x^I$, for increasing multi-indices $I = (i_0, \dots, i_{a-1})$. Then we can compute

$$
\begin{aligned}
\mathrm{d}(\mathrm{d}\omega) &= \mathrm{d}\left(\mathrm{d}\left(\sum_I \omega_I \;\mathrm{d}x^I\right)\right) \\
&= \mathrm{d}\left(\sum_I \mathrm{d}\omega_I \wedge \mathrm{d}x^I\right) \\
&= \mathrm{d}\left(\sum_I \sum_j \frac{\partial \omega_I}{\partial x^j} \;\mathrm{d}x^j \wedge \mathrm{d}x^I\right) \\
&= \sum_I \sum_j \mathrm{d}\frac{\partial \omega_I}{\partial x^j} \wedge \mathrm{d}x^j \wedge \mathrm{d}x^I \\
&= \sum_I \sum_j \sum_k \frac{\partial^2 \omega_I}{\partial x^j\, \partial x^k} \;\mathrm{d}x^k \wedge \mathrm{d}x^j \wedge \mathrm{d}x^I
\end{aligned}
$$

Now, we can see that the factor $\sum_j \sum_k \frac{\partial^2 \omega_I}{\partial x^j\, \partial x^k} \;\mathrm{d}x^k \wedge \mathrm{d}x^j$ vanishes because the wedge product $\mathrm{d}x^k \wedge \mathrm{d}x^j$ is antisymmetric, while the partial derivative $\frac{\partial^2 \omega_I}{\partial x^j\, \partial x^k}$ is symmetric. We conclude that $\mathrm{d}(\mathrm{d}\omega) = 0$. □

(d) Given $F : U \to V$ a smooth map, $\mathrm{d}(F^*\omega) = F^* \,\mathrm{d}\omega$.

Solution. Let $\omega \in \Omega^p(M)$ be a smooth p-form on a manifold M. Let us show that for any smooth map $F : N \to M$ we have the identity $\mathrm{d}(F^*\omega) = F^*(\mathrm{d}\omega)$.

To show that the identity holds at some point $p \in N$, we take a chart (U, φ) of M that covers the point $F(p)$. Over the set U, the form ω can be written as

$$
\omega = \sum_{i_0 < \dots < i_{p-1}} \omega_{i_0, \dots, i_{p-1}} \;\mathrm{d}\varphi^{i_0} \wedge \ldots \wedge \mathrm{d}\varphi^{i_{p-1}}
$$

Then on the set $F^{-1}(U)$ we have

$$F^*\omega = \sum_{i_0<\dots<i_{p-1}} F^*\omega_{i_0,\dots,i_{p-1}} F^*(\mathrm{d}\varphi^{i_0}) \wedge \dots \wedge F^*(\mathrm{d}\varphi^{i_{p-1}})$$
$$= \sum_{i_0<\dots<i_{p-1}} F^*\omega_{i_0,\dots,i_{p-1}}\, \mathrm{d}(F^*\varphi^{i_0}) \wedge \dots \wedge \mathrm{d}(F^*\varphi^{i_{p-1}})$$

Here we have used the fact that the pullback operator commutes with sums and wedge products (which follows easily from the definitions) and the identity $F^*(\mathrm{d}h) = \mathrm{d}(F^*h)$ for the case that $h \in \mathcal{C}^\infty(M)$ (which is equivalent to the chain rule for the composite function $h \circ F$). Using these same facts we can finish the computation as follows. Applying the differential operator, we get

$$\mathrm{d}(F^*\omega) = \sum_{i_0<\dots<i_{p-1}} \mathrm{d}(F^*\omega_{i_0,\dots,i_{p-1}}) \wedge \mathrm{d}(F^*\varphi^{i_0}) \wedge \dots \wedge \mathrm{d}(F^*\varphi^{i_{p-1}})$$
$$(\text{ there are no more terms since } \mathrm{d}\,\mathrm{d}(F^*\varphi^i) = 0 \text{ for all } i)$$
$$= \sum_{i_0<\dots<i_{p-1}} F^*(\mathrm{d}\omega_{i_0,\dots,i_{p-1}}) \wedge F^*(\mathrm{d}\varphi^{i_0}) \wedge \dots \wedge F^*(\mathrm{d}\varphi^{i_{p-1}})$$
$$= F^*\left(\sum_{i_0<\dots<i_{p-1}} \mathrm{d}\omega_{i_0,\dots,i_{p-1}} \wedge \mathrm{d}\varphi^{i_0} \wedge \dots \wedge \mathrm{d}\varphi^{i_{p-1}}\right)$$
$$= F^*(\mathrm{d}\omega).$$

This shows that the identity $\mathrm{d}(F^*\omega) = F^*(\mathrm{d}\omega)$ holds on the set $F^{-1}(U)$, and in particular, at the point p. Since the point $p \in N$ is arbitrary, we conclude that the identity holds on the whole manifold N. □

Exercise 12.3. Suppose that M is a smooth manifold which is the union of two orientable open sub-manifolds with connected intersection. Show that then M is orientable. Use this to prove that $\mathbb{S}^n$ is orientable.

Solution. Let $M = U_0 \cup U_1$, where both sets U_i are open an orientable, and the intersection $U_0 \cap U_1$ is connected. Let us show that M is orientable as well. Let $\mathcal{O}^0$, $\mathcal{O}^1$ be respective orientations of the submanifolds U_0, U_1. By reversing the sign of $\mathcal{O}^1$ if necessary, we can ensure that both orientation fields $\mathcal{O}^i$ coincide at some point of the set $U_0 \cap U_1$, and it follows that they coincide at all points the set $U_0 \cap U_1$, since this set is open and connected. Thus we can define an orientation field $\mathcal{O}$ on M by the formula

$$\mathcal{O}_q = \begin{cases} \mathcal{O}^0_q \text{ if } q \in U_0 \\ \mathcal{O}^1_q \text{ if } q \in U_1 \end{cases}$$

and it is clear that this orientation field is continuous on M since it is continuous on each of the sets U_i. Therefore $\mathcal{O}$ is an orientation on M, which proves that M is an orientable manifold.

For the case of the sphere $\mathbb{S}^n$ (for $n \geq 2$), denoting P_0 and P_1 the north and south poles, the stereographic charts show that each open set $U_i = \mathbb{S}^n \setminus P_i$ is diffeomorphic to $\mathbb{R}^n$ and therefore orientable. The intersection $U_0 \cap U_1$ is connected. (In fact, it is homotopically equivalent to the Equator, which is an $(n-1)$-sphere. Here we use the fact that $n \geq 2$.) Thus by the theorem above, the sphere $\mathbb{S}^n$ is orientable.

To show that $\mathbb{S}^1$ is orientable we use the fact that the n-torus $\mathbb{T}^n$ is orientable for all n.

Another way to show that $\mathbb{S}^n$ is orientable is by proving the following theorem: If M is an orientable manifold, then any regular level set $h^{-1}(c)$ of a function $h \in \mathcal{C}^\infty(M)$ is an orientable hypersurface. Another (similar) theorem says that an embedded hypersurface of an orientable manifold is orientable if (and only if!) it admits a transverse vector field. □

Exercise 12.4. * Let M a smooth manifold. Show that $\mathrm{T}M$ and T^*M are always orientable manifolds.

Solution. We will just show that $\mathrm{T}M$ is orientable. (The case of T^*M can be treated similarly.) We first recall the natural smooth structure of the manifold $\mathrm{T}M$. Denote $\pi : \mathrm{T}M \to M$ the natural projection that maps each $(p, v) \in TM$ to the point $p \in M$.

Local parametrizations of $\mathrm{T}M$: For each smooth local parametrization $\varphi : \widetilde{U} \to U$ of M (where $\widetilde{U} \subseteq \mathbb{R}^n$, $U \subseteq M$ are open sets) we define a local parametrization $\Phi : \widetilde{U} \times \mathbb{R}^n \to \pi^{-1}(U)$ of $\mathrm{T}M$ by the formula

$$\Phi(x, v) = (\varphi(x), D_x\varphi(v)).$$

The local parametrizations obtained in this way form a smooth parametrization atlas for $\mathrm{T}M$.

Transition maps: Suppose $\psi : \widetilde{V} \to V$ is another local parametrization of M and $\Psi : \widetilde{V} \times \mathbb{R}^n \to \pi^{-1}(V)$ is the corresponding local parametrization of $\mathrm{T}M$, defined by the formula $\Psi(y, w) = (\psi(y), D_y\psi(w))$. Then the transition map from Φ to Ψ is the map $\Psi^{-1} \circ \Phi : \varphi^{-1}(U \cap V) \times \mathbb{R}^n \to \psi^{-1}(U \cap V) \times \mathbb{R}^n$ given by

$$\Psi^{-1} \circ \Phi(x, v) = (\psi^{-1} \circ \varphi(x), \, \mathrm{D}_x(\psi^{-1} \circ \varphi)(v)) =: (y, w).$$

Now let us show that $\mathrm{T}M$ is orientable. We claim that there exists an orientation of the manifold $\mathrm{T}M$ such that the local parametrizations of $\mathrm{T}M$ defined above are positive (i.e orientation preserving). For this, it suffices to show that for any pair of local parametrizations Φ, Ψ as defined above, the differential $\mathrm{D}_{(x,v)}(\Psi^{-1} \circ \Phi)$ of the transition map at any point $(x, v) \in \mathrm{Dom}(\Psi^{-1} \circ \Phi) = \varphi^{-1}(U \cap V) \times \mathbb{R}^n$ has positive determinant. (See Proposition 7.1.8 of the lecture notes.)

The differential $\mathrm{D}_{(x,v)}(\Psi^{-1} \circ \Phi)$ is represented in standard coordinates by a $2n \times 2n$ matrix

$$[\,\mathrm{D}_{(x,v)}(\Psi^{-1} \circ \Phi)] = \begin{pmatrix} \left(\frac{\partial y^j}{\partial x^i}\right)_{i,j} & \left(\frac{\partial y^j}{\partial v^i}\right)_{i,j} \\ \left(\frac{\partial w^j}{\partial x^i}\right)_{i,j} & \left(\frac{\partial w^j}{\partial v^i}\right)_{i,j} \end{pmatrix} = \begin{pmatrix} [\,\mathrm{D}_x(\psi^{-1} \circ \varphi)] & 0 \\ * & [\,\mathrm{D}_x(\psi^{-1} \circ \varphi)] \end{pmatrix},$$

which has positive determinant

$$\det[\,\mathrm{D}_{(x,v)}(\Psi^{-1} \circ \Phi)] = (\det[\,\mathrm{D}_x(\psi^{-1} \circ \varphi)])^2 > 0,$$

as needed. □

Recall that a smooth covering map $\pi : E \to M$ is a smooth surjective morphism such that for each $p \in M$ there exists a open neighbourhood U in M with $\pi^{-1}(U) = \coprod_{i=1}^k V_i$ with $\pi|_{V_i} : V_i \to U$ is a diffeomorphism.

The following two results explain how orientations behave with respect to smooth covering maps and give us criteria to show when a manifold is not orientable.

Theorem Let E be a connected, oriented, smooth manifold and $\pi : E \to M$ be a smooth normal covering map. [1] Then M is orientable if and only if the action of $\mathrm{Aut}_\pi(E)$ is orientation preserving

Exercise 12.5. * Using the above theorem, prove that $\mathbb{RP}^n$ is orientable if and only if n is odd.

Hint: Consider $q : \mathbb{S}^n \to \mathbb{RP}^n$ the quotient map. Prove that this is a normal smooth covering map. Then prove that the only non trivial automorphism of q is the antipodal map $x \mapsto -x$ which is orientation preserving if and only if n is odd.

[1] For us normal will just means that the group $\mathrm{Aut}_\pi(E)$ of automorphism of E commuting with π acts transitively on the fiber.

Solution. Let $A : x \mapsto -x$ be the antipodal map on $\mathbb{R}^{n+1}$. This map is linear and represented by the identity matrix $-I_{n+1}$, which has determinant $(-1)^{n+1}$. Therefore the map A preserves the (standard) orientation of $\mathbb{R}^{n+1}$ if and only if n is odd.

Now we must show that the antipodal map resticted to the sphere $\mathbb{S}^n$, which we denote by $\alpha : \mathbb{S}^n \to \mathbb{S}^n$, behaves in a similar way: it preserves the orientation of the sphere if and only if n is odd. For this we use the following orientation $\mathcal{O}$ of the sphere: for any point $p \in \mathbb{S}^n$ and any basis $V_0, \dots, V_{n-1}$ of $\mathrm{T}_p(\mathbb{S}^n) \subseteq \mathbb{R}^{n+1}$, we have

$$\mathcal{O}_p(V_0, \dots, V_{n-1}) = \mathcal{O}^{std}(p, V_0, \dots, V_{n-1})$$

where $\mathcal{O}^{std}$ is the standard orientation of $\mathbb{R}^{n+1}$. This orientation field $\mathcal{O}$ is continuous since it can be written as

$$\mathcal{O}^{std}(p, V_0, \dots, V_{n-1}) = \operatorname{sgn}\det(p, V_0, \dots, V_{n-1}),$$

thus if we take a chart (U, φ) of $\mathbb{S}^n$, we see that the function

$$\begin{aligned} p \mapsto \ \mathcal{O}_p\left(\frac{\partial}{\partial\varphi^0}\bigg|_p, \dots, \frac{\partial}{\partial\varphi^{n-1}}\bigg|_p\right) &= \mathcal{O}^{std}\left(p, \frac{\partial}{\partial\varphi^0}\bigg|_p, \dots, \frac{\partial}{\partial\varphi^{n-1}}\bigg|_p\right) \\ &= \operatorname{sgn}\det\left(p, \frac{\partial}{\partial\varphi^0}\bigg|_p, \dots, \frac{\partial}{\partial\varphi^{n-1}}\bigg|_p\right) = \pm 1 \end{aligned}$$

is locally constant on U.

Now let us show that α preserves the orientation $\mathcal{O}$ if and only if n is odd. For any point $p \in \mathbb{S}^n$ and vectors $V_i \in \mathrm{T}_p\mathbb{S}^n$ as above, we have

$$\begin{aligned} \mathcal{O}_{\alpha(p)}(D_p\alpha(V_0), \dots, D_p\alpha(V_{n-1})) &= \mathcal{O}^{std}(\alpha(p), D_p\alpha(V_0), \dots, D_p\alpha(V_{n-1})) \\ &= \mathcal{O}^{std}(-p, -V_0, \dots, -V_{n-1}) \\ &= (-1)^{n+1}\, \mathcal{O}^{std}(p, V_0, \dots, V_{n-1}) \\ &= (-1)^{n+1}\, \mathcal{O}_p(V_0, \dots, V_{n-1}) \\ &= \mathcal{O}_p(V_0, \dots, V_{n-1}) \quad \text{ if and only if } n \text{ is odd.} \end{aligned}$$

To finish, we apply the theorem mentioned above to the covering map $\pi : \mathbb{S}^n \to \mathbb{RP}^n$. The manifold $\mathbb{S}^n$ is connected and oriented. The automorphism group of the covering map π is $\{\mathrm{id}_{\mathbb{S}^n}, \alpha\}$, and it is normal since it acts transitively on the fibers, which are of the form $\{p, -p\}$ for $p \in \mathbb{S}^n$. The identity map clearly preserves the orientation, and the map α does if and only if n is odd. Therefore, by the above theorem, the projective space $\mathbb{RP}^n$ is orientable if and only if n is odd. $\square$