APA complement: Monotone class theorem

An event $\underline{\underline{A}}$ is independent of $\sigma(X_1, \dots, X_n)$ $\forall n \geq 1$

This implies that A is independent of $\sigma(X_n, n \geq 1)$?

Monotone class theorem

If $\mathcal{G}$ is an algebra & $\mathcal{M}$ is a monotone class

& $\mathcal{G} \subset \mathcal{M}$, then $\sigma(\mathcal{G}) \subset \mathcal{M}$

<u>Algebra:</u> $\mathcal{G}$ is an algebra if

- $\emptyset, \Omega \in \mathcal{G}$
- if $A \in \mathcal{G}$ then $A^c \in \mathcal{G}$
- if $A_1 .. A_n \in \mathcal{G}$ then $\bigcup_{j=1}^{n} A_j \in \mathcal{G}$

<u>Monotone class:</u> $\mathcal{M}$ is a monotone class if

- if $A_1 \subset A_2 \subset \ldots \in \mathcal{M}$, then $\bigcup_{n \geq 1} A_n \in \mathcal{M}$
- if $B_1 \supset B_2 \supset \ldots \in \mathcal{M}$, then $\bigcap_{n \geq 1} B_n \in \mathcal{M}$

$$\mathcal{G} = \bigcup_{n \geq 1} \underbrace{\sigma(X_1, \dots, X_n)}_{= \mathcal{F}_n \ \sigma\text{-field}} \quad \text{is an algebra}$$

$$\mathcal{M} = \{ B \in \mathcal{F} : \mathbb{P}(B \cap A) = \mathbb{P}(B) \cdot \mathbb{P}(A) \}$$

= collection of events in $\mathcal{F}$ which are indep. of A

$\mathcal{M}$ is a monotone class:

Take $B_1 \supset B_2 \supset \dots \in \mathcal{M}$

$$\mathbb{P}\Big(\bigcap_{n \geq 1} B_n \cap A \Big) = \lim_{n \to \infty} \mathbb{P}(B_n \cap A) = \lim_{n \to \infty} \mathbb{P}(B_n) \cdot \mathbb{P}(A)$$

$$= \mathbb{P}\Big(\bigcap_{n \geq 1} B_n \Big) \cdot \mathbb{P}(A)$$

Assumption:

- A is independant of $\mathcal{F}_n$ $\forall n \geq 1$

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B) \quad \forall B \in \mathcal{F}_n$$

- A is "independant" of $\bigcup_{n \geq 1} \mathcal{F}_n$ (!not necessarily a σ-field)

because $\forall B \in \bigcup_{n \geq 1} \mathcal{F}_n$, $\exists N \geq 1$ st $B \in \mathcal{F}_N$

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B) \quad \forall B \in \underline{\underline{\bigcup_{n \geq 1} \mathcal{F}_n}} = \mathcal{G}$$

i.e. $\mathcal{G} \subset \mathcal{M}$

- By the monotone class thm, $\underbrace{\sigma(\mathcal{G})}_{= \sigma(X_n, n \geq 1)} \subset \mathcal{M}$