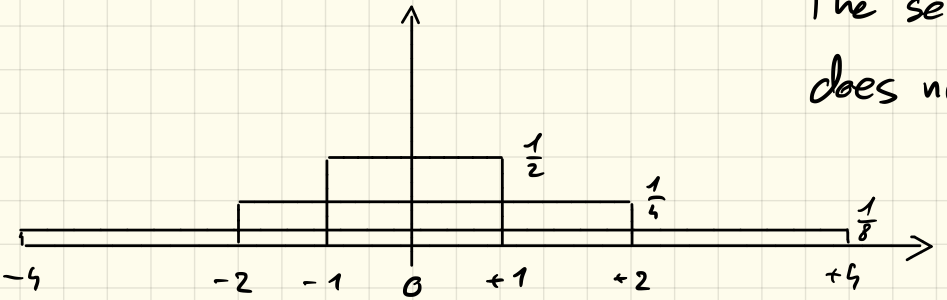
APA complement: Convergence in distribution

Let $(X_n, n \geq 1)$ be a sequence of r.v.'s such that $X_n \sim \mathcal{U}([-n, +n])$ $\forall n \geq 1$.



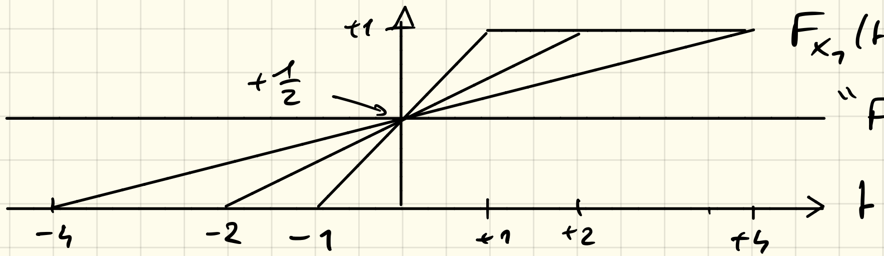
The sequence $(X_n, n \geq 1)$ does not converge in distribution.

Uniform on $\mathbb{R}$? $\quad p_X(x) = p \ \forall x \in \mathbb{R}$, $\int_{\mathbb{R}} \underbrace{p_X(x)}_{p} dx = 1$

$\cancel{p > 0}$

$\cancel{p = 0}$

1) Look at the cdfs: $F_{X_n}(t)$



$$F_{X_n}(t) = \begin{cases} 0 & \text{if } t < -n \\ \frac{t+n}{2n} & \text{if } -n \leq t \leq +n \\ 1 & \text{if } t > +n \end{cases}$$

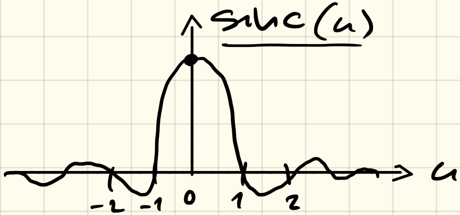
fix t: $\frac{t+n}{2n} \xrightarrow[n\to\infty]{} \frac{1}{2}$

we have $\lim_{n\to\infty} F_{X_n}(t) = \underbrace{\frac{1}{2}}_{\text{not a cdf}} \quad \forall t \in \mathbb{R}$

2) look at the <u>characteristic functions</u>:

$$\phi_{X_n}(t) = \mathbb{E}(e^{itX_n}) = \frac{1}{2n}\int_{-n}^{+n} e^{itx}\,dx$$

$$= \begin{cases} \dfrac{e^{itn} - e^{-itn}}{2n \cdot it} = \dfrac{\sin(tn)}{tn} = \operatorname{sinc}(tn) & (\text{if } t \neq 0) \\ 1 & (\text{if } t = 0) \end{cases}$$



<u>fix t</u>: $\lim_{n\to\infty} \phi_{X_n}(t) = \lim_{n\to\infty} \operatorname{sinc}(tn)$

$$= \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

but "$\phi_\infty(t)$" $= \begin{cases} 1 & \text{if } t=0 \\ 0 & \text{if } t \neq 0 \end{cases}$ is not a characteristic fn (because not continuous in $t=0$)