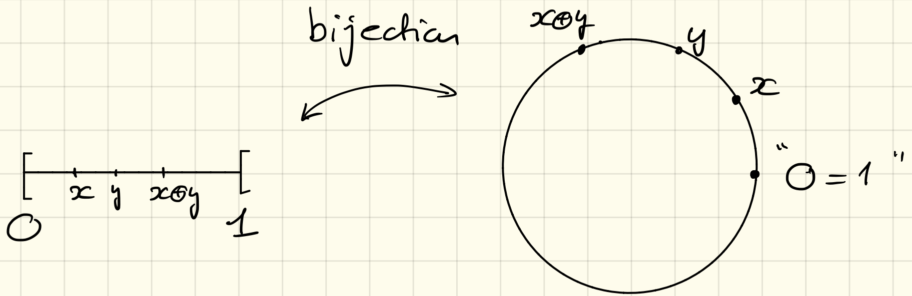
APA complement

1°) Existence of non-measurable sets:

Consider the set $\Omega = [0,1[$ with the "addition modulo 1"

i.e. $x \oplus y$ is the decimal part of the sum $x+y$;

Ω can be seen as a flat representation of a circle:



On Ω, consider the following equivalence relation:

$$x \sim y \text{ if } \exists\, r \in \mathbb{Q} \cap \Omega \quad \text{such that } x = y \oplus r$$

This (strange) equivalence relation allows us to separate the set Ω into disjoint equivalence classes. Define now A to be the set constructed with a representative of each equivalence class (the axiom of choice guarantees that such a set exists).

For $r \in \mathbb{Q} \cap \Omega$, define $A_r = A \oplus r = \{ x \oplus r : x \in A \}$ and it holds that $A_r \cap A_s = \phi$ for $r \neq s$, $r, s \in \mathbb{Q} \cap \Omega$ and $\Omega = \bigcup_{r \in \mathbb{Q} \cap \Omega} A_r$ is therefore a countable & disjoint union.

We also want to define a probability measure $\mathbb{P}$ on $\Omega = [0,1[$ such that $\mathbb{P}(B) = \mathbb{P}(B \oplus z)$ for every measurable set $B \subset \Omega$ and $z \in \Omega$

This is the translation invariance of the Lebesgue measure on Ω (or the rotation invariance if Ω is viewed as a circle).

Problem: By the axioms of probability, we should have
$$\underbrace{\mathbb{P}(\Omega)}_{=1} = \sum_{r \in \mathbb{Q} \cap \Omega} \mathbb{P}(A_r) = \sum_{r \in \mathbb{Q} \cap \Omega} \mathbb{P}(A)$$
and this is impossible (both $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) > 0$ lead to a contradiction). Conclusion: A is not measurable #

2°) <u>Any cdf has at most a countable number of jumps:</u>

F makes a jump at position $t \in \mathbb{R}$ if $F(t) - \underbrace{F(t^-)}_{= \lim_{\varepsilon \downarrow 0} F(t - \varepsilon)} > 0$

Let us now consider $A = \{ t \in \mathbb{R} : F(t) - F(t^-) > 0 \}$ and show that A is countable. Define

$$A_n = \left\{ t \in \mathbb{R} : \frac{1}{n+1} < F(t) - F(t^-) \leq \frac{1}{n} \right\} \text{ for } n \geq 1$$

Observe that each A_n is finite (more precisely, $|A_n| \leq n+1$) and $A = \bigcup_{n \geq 1} A_n$. Therefore, A is countable. #