APA lecture 1

σ-fields

Ω = "fundamental" set (any set)

Ex: $\Omega = \{1, 2, 3, 4, 5, 6\}$ die roll

$\Omega = \mathbb{R}$, $\Omega = [0, 1]$

$\Omega = \{$ orange, banana, apple $\}$

Def:

A σ-field on Ω is a collection $\mathcal{F}$ of subsets of Ω s.t.:
= "events"

(i) $\emptyset \in \mathcal{F}$, $\Omega \in \mathcal{F}$

(ii) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

(iii) if $(A_n, n \geq 1) \subset \mathcal{F}$, then $\bigcup_{n \geq 1} A_n \in \mathcal{F}$

<u>Example:</u> • $\Omega = \{1, \dots, 6\}$ (die roll experiment)

$\mathcal{F}_0 = \{\emptyset, \Omega\}$ trivial σ-field ($\Leftrightarrow$ no information)

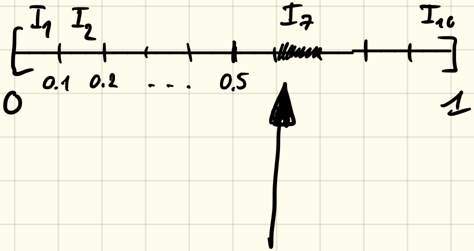
$\mathcal{F}_1 = \{\emptyset, \{1\}, \{2 .. 6\}, \Omega\}$

$\mathcal{F}_2 = \{\emptyset, \{1\,3\,5\}, \{2\,4\,6\}, \Omega\}$

$\mathcal{F} = \mathcal{P}(\Omega) = \{$ all subsets of $\Omega \}$ complete σ-field

$= \{\emptyset, \{1\}, \{2\}, \dots, \{6\}, \{1,2\}, \dots, \{1,2,3\}, \dots \Omega\}$

• $\Omega = [0, 1]$ (measurement of some concentration)



$\Omega = I_1 \cup \dots \cup I_n \qquad I_j \cap I_k = \emptyset \quad \forall j \neq k$

$\mathcal{F} = \{\emptyset, I_1, I_2 \dots I_n,$

$I_1 \cup I_2, \dots, I_1 \cup I_2 \cup I_3, \dots, \Omega\}$

σ-field generated by a collection of events

<u>Def:</u> Let $\mathcal{A} = \{A_i, i \in I\}$ be a collection of subsets of Ω
The σ-field generated by $\mathcal{A}$ is the smallest collection of subsets of Ω which contains all elements of $\mathcal{A}$ and is itself a σ-field.

<u>Notation:</u> $\sigma(\mathcal{A})$

<u>Ex:</u> $\Omega = \{1, \ldots, 6\}$

atoms

$\mathcal{A}_1 = \{\{1\}\}$ $\quad \sigma(\mathcal{A}_1) = \mathcal{F}_1 = \{\phi, \{1\}, \{2 \ldots 6\}, \Omega\}$

$\mathcal{A}_2 = \{\{1 3 5\}\}$ $\quad \sigma(\mathcal{A}_2) = \mathcal{F}_2 = \{\phi, \{1 3 5\}, \{2 4 6\}, \Omega\}$

$\mathcal{A} = \{\{\underline{1 2} 3\}, \{\underline{1 3 5}\}\}$ $\quad \sigma(\mathcal{A}) = \ldots$

atoms

($\rightarrow$ exercise)

$$A = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\} \qquad \sigma(A) = P(\Omega)$$

6 "atoms" $\qquad$ 2^6 elements

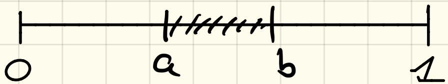
- $\underline{\Omega = [0, 1]} = I_1 \cup \dots \cup I_n \qquad I_j \cap I_k = \emptyset \quad \forall j \neq k$

  disjoint

$$\mathcal{F} = \sigma(I_1, \dots, I_n)$$

n atoms of $\mathcal{F}$

- $\Omega = [0, 1]$:



$0 \le a < b \le 1$

$$\mathcal{F} = \sigma\left(\underline{]a, b[}, \ 0 \le a < b \le 1, \ \{0\}, \{1\} \right) := \mathcal{B}([0,1])$$

= Borel σ-field on $[0,1]$

(= <u>definition</u>)

$= P([0,1])$? $\underline{\underline{\text{no}}}$

What sets are contained in $\mathcal{B}([0,1])$?

$$\{x\} = \underbrace{\bigcap_{n\geq 1} \underbrace{\left]x-\tfrac{1}{n}, x+\tfrac{1}{n}\right[}_{\in \mathcal{B}([0,1])}}_{\in \mathcal{B}([0,1])} \quad \forall x \in [0,1] \implies \{x\} \in \mathcal{B}([0,1])$$

Indeed: for any σ-field $\mathcal{F}$, we have the following:

- If $(A_n, n\geq 1) \subset \mathcal{F}$, then $\bigcap_{n\geq 1} A_n \in \mathcal{F}$
- If $A, B \in \mathcal{F}$, then $B \setminus A \in \mathcal{F}$

Pf: $\cdot$ $\bigcap_{n\geq 1} A_n = \left(\bigcup_{n\geq 1} A_n^c\right)^c \in \mathcal{F}$

$\cdot$ $B \setminus A = B \cap A^c \in \mathcal{F}$ ✓

- Therefore, any $[a,b] = \{a\} \cup]a,b[\cup \{b\} \in \mathcal{B}([0,1])$ and <u>many</u> more sets $\in \mathcal{B}([0,1])$

- $\mathcal{B}([0,1])$ is <u>not</u> generated by its atoms!

  Indeed, its atoms are: $\{x\}, x \in [0,1]$, but

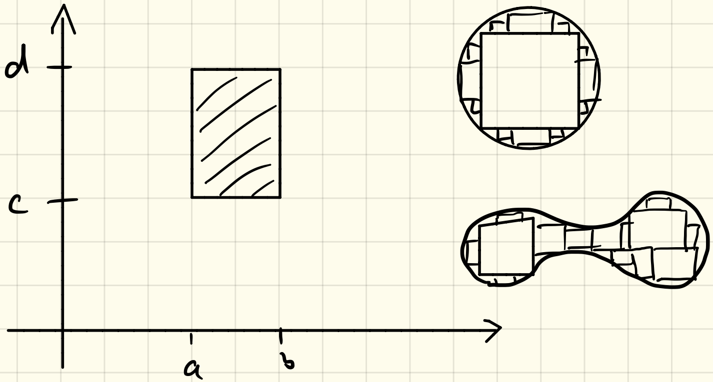
$$\sigma(\{\{x\}, x \in (0,1]\}) \neq \mathcal{B}([0,1])$$

$=$ something much smaller! ($\longrightarrow$ exercise)

Examples: (= definitions)

- $\Omega = \mathbb{R}$: $\mathcal{B}(\mathbb{R}) = \sigma\left(\{\,]a,b[\,,\ a<b \}\right) \neq \mathcal{P}(\mathbb{R})$

- $\Omega = \mathbb{R}^2$: $\mathcal{B}(\mathbb{R}^2) = \sigma\left(\{\,]a,b[\times]c,d[\,,\ a<b,\ c<d \}\right)$
  $\neq \mathcal{P}(\mathbb{R}^2)$



Sub-σ-fields

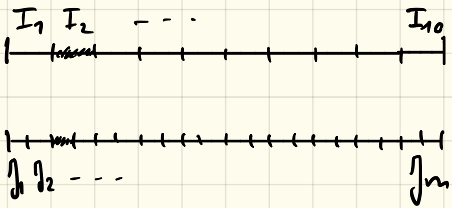
Def: Let Ω be a set and $\mathcal{F}$ a σ-field on Ω.

A <u>sub-σ-field</u> of $\mathcal{F}$ is a collection $\mathcal{G}$ of subsets of Ω

s.t.:
- if $A \in \mathcal{G}$, then $A \in \mathcal{F}$
- $\mathcal{G}$ is itself a σ-field

Notation: $\mathcal{G} \subset \mathcal{F}$

Ex: $\Omega = [0, 1]$



$\rightarrow \mathcal{G} = \sigma(I_1 \dots I_n)$

$\rightarrow \mathcal{F} = \sigma(J_1 \dots J_m)$

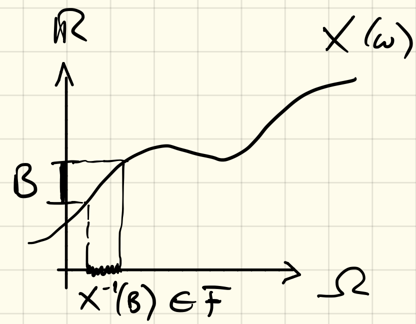
$\mathcal{G} \subset \mathcal{F}$

⚠
- if $A \in \mathcal{G}$ and $\mathcal{G} \subset \mathcal{F}$, then $A \in \mathcal{F}$
- if $A \subset B$ and $B \in \mathcal{F}$, then this does <u>not</u> imply $A \in \mathcal{F}$

Random variables

Def: A random variable on $(\Omega, \mathcal{F})$ is a function
$X : \Omega \to \mathbb{R}$ satisfying

$$\underbrace{\{\omega \in \Omega : X(\omega) \in B\}}_{= \{X \in B\} = X^{-1}(B)} \in \mathcal{F} \qquad \forall B \in \mathcal{B}(\mathbb{R})$$



Terminology: (equivalent)

X is said to be an $\mathcal{F}$-measurable random variable.

Rmk:
- if $\mathcal{F} = \mathcal{P}(\Omega)$, then every function X is $\mathcal{F}$-measurable
- if $\mathcal{F} = \{\emptyset, \Omega\}$, then only constant functions X are $\mathcal{F}$-measurable

Proposition

X is an $\mathcal{F}$-measurable r.v. iff

$$\{\omega \in \Omega : X(\omega) \leq t\} \in \mathcal{F} \qquad \forall t \in \mathbb{R}$$

Examples

- if $A \in \mathcal{F}$, then $X(\omega) = 1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases}$

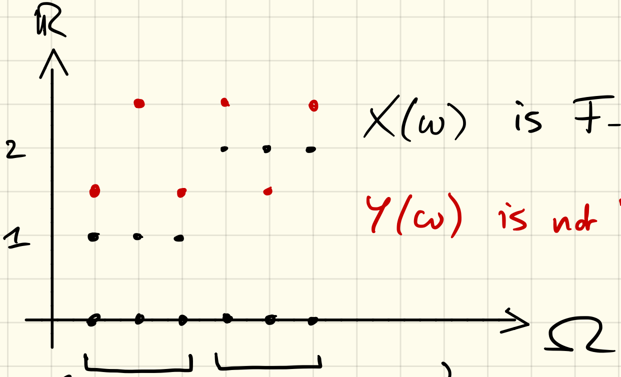
  is an $\mathcal{F}$-measurable r.v.

- $\Omega = \{1..6\}$ $\quad \mathcal{F}_1 = \{\emptyset, \{1\}, \{2...6\}, \Omega\}$ $\quad \mathcal{F}_2 = \{\emptyset, \{135\}, \{246\}, \Omega\}$

  $X_a(\omega) = \begin{cases} 1 & \text{if } \omega = 1 \\ 2 & \text{if } \omega > 1 \end{cases}$ is $\mathcal{F}_1$-measurable, is not $\mathcal{F}_2$-measurable

  $X_b(\omega) = \omega$ is not $\mathcal{F}_1$-measurable, is not $\mathcal{F}_2$-measurable

  $X_c(\omega) = \begin{cases} -1 & \text{if } \omega \text{ is odd} \\ +1 & \text{if } \omega \text{ is even} \end{cases}$ is not $\mathcal{F}_1$-measurable, is $\mathcal{F}_2$-measurable

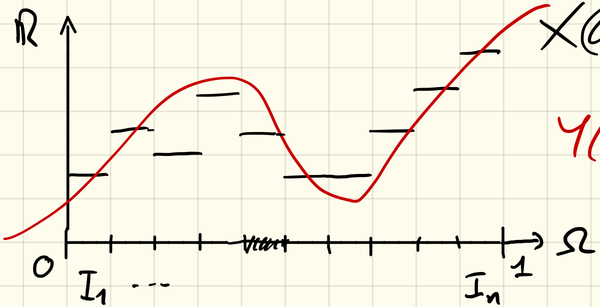


$X(\omega)$ is $\mathcal{F}$-measurable

$Y(\omega)$ is not $\mathcal{F}$-measurable

$\mathcal{F} = \{\phi, A, A^c, \Omega\}$

<u>Ex</u>: $\Omega = [0,1]$



$X(\omega)$ is $\mathcal{F}$-measurable

$Y(\omega)$ is not $\mathcal{F}$-measurable

$\mathcal{F} = \sigma(I_1, \ldots, I_n)$

Borel-measurable functions

Def: A Borel-measurable function is a function $f: \mathbb{R} \to \mathbb{R}$ such that

$$\underbrace{\{x \in \mathbb{R} : f(x) \in B\}}_{= \{f \in B\} = f^{-1}(B)} \in \mathcal{B}(\mathbb{R}) \quad \forall B \in \mathcal{B}(\mathbb{R})$$

Proposition: (without proof)

If $f: \mathbb{R} \to \mathbb{R}$ is continuous, then it is Borel-measurable.

Proposition:

If $f: \mathbb{R} \to \mathbb{R}$ is Borel-measurable and X is an $\mathcal{F}$-measurable r.v., then $Y = f(X)$ is also an $\mathcal{F}$-measurable r.v.

Proof: Let $B \in \mathcal{B}(\mathbb{R})$:

$$\{\omega \in \Omega : Y(\omega) \in B\} = \{\omega \in \Omega : f(X(\omega)) \in B\}$$

$$= \{\omega \in \Omega : X(\omega) \in \underbrace{f^{-1}(B)}_{\in \mathcal{B}(\mathbb{R}) \text{ because } f \text{ is Borel-measurable}}\} \in \mathcal{F}$$ because X is an $\mathcal{F}$-measurable r.v.

#

σ-field generated by a collection of random variables

Def: Let Ω be a set and $\mathcal{F}$ be a σ-field on Ω

Let $(X_i, i \in I)$ be a collection of $\mathcal{F}$-measurable r.v.'s

The σ-field generated by $(X_i, i \in I)$, denoted as

$\sigma(X_i, i \in I)$ is the smallest σ-field $\mathcal{G}$ on Ω such

that all random variables X_i are $\mathcal{G}$-measurable.

Proposition:

$$\sigma(X_i, i \in I) = \sigma\left(\{X_i \in B\},\ i \in I,\ B \in \mathcal{B}(\mathbb{R})\right)$$

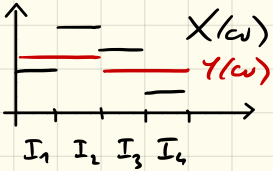
$$= \sigma\left(\{X_i \leq t\},\ i \in I,\ t \in \mathbb{R}\right)$$

<u>Ex</u>: • if $X_0(\omega) = c$ $\forall \omega \in \Omega$, then $\sigma(X_0) = \mathcal{F}_0 = \{\phi, \Omega\}$

• $\Omega = \{1..6\}$ and $X(\omega) = \omega$, then $\sigma(X) = \mathcal{P}(\Omega)$

$Y(\omega) = \begin{cases} -1 & \text{if } \omega \text{ is odd} \\ +1 & \text{if } \omega \text{ is even} \end{cases}$, then $\sigma(Y) = \{\phi, \{135\}, \{246\}, \Omega\}$

• $\Omega = [0, 1]$



$\sigma(X) = \sigma(I_1, I_2, I_3, I_4)$

$\sigma(Y) = \sigma(I_1 \cup I_2, I_3 \cup I_4)$

<u>Proposition</u>:

If X is an $\mathcal{F}$-measurable r.v. & $f: \mathbb{R} \to \mathbb{R}$ is Borel-measurable, then $Y = f(X)$ is a $\sigma(X)$-measurable r.v.

And every r.v. Y which is $\sigma(X)$-measurable may be written as $Y = f(X)$ for some Borel-measurable fn f.

So if $Y = f(X)$, then $\sigma(Y) \subset \sigma(X)$.

& if f is invertible, then $X = f^{-1}(Y)$ and $\sigma(X) \subset \sigma(Y)$

and so in this case $\sigma(X) = \sigma(Y)$.

But in general, this might not be the case: for ex., take

$Y = f(X^2)$; then the information about the sign of X is lost

in $\sigma(Y)$.

If $Y = f(X_1, X_2)$ where $f: \mathbb{R}^2 \to \mathbb{R}$ is Borel-measurable,

then $\sigma(Y) \subset \sigma(X_1, X_2)$.