APA lecture 12

Martingales

Def: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space

A filtration is a sequence of sub-σ-fields of $\mathcal{F}$ $(\mathcal{F}_n, n \in \mathbb{N})$

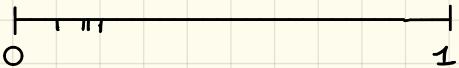
such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$, $\forall n \in \mathbb{N}$.

Ex: $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, $\mathbb{P}$ = Lebesgue measure on $[0,1]$

$X_n(\omega) = n^{th}$ decimal of $\omega \in [0,1]$ $\quad n \geq 1$

$\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \sigma(X_1)$, $\mathcal{F}_2 = \sigma(X_1, X_2)$ ... $\mathcal{F}_n = \sigma(X_1 .. X_n)$ $\quad n \geq 1$

$\omega = 0{,}17392\ldots$



Def: Let $(\mathcal{F}_n, n \in \mathbb{N})$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$

A square-int. martingale w.r.t $(\mathcal{F}_n, n \in \mathbb{N})$ is a discrete-time stochastic process $(M_n, n \in \mathbb{N})$ such that

(i) $\mathbb{E}(M_n^2) < +\infty \quad \forall n \in \mathbb{N}$

(ii) M_n is $\mathcal{F}_n$-measurable $\forall n \in \mathbb{N}$ (adapted process)

$\longrightarrow$ (iii) $\mathbb{E}(M_{n+1} \mid \mathcal{F}_n) = M_n$ a.s. $\quad \forall n \in \mathbb{N}$

Rmk: Because $\mathbb{E}(M_{n+1} \mid \mathcal{F}_n)$ is $\mathcal{F}_n$-measurable by definition, condition (ii) is actually redundant.

Rmk: martingale $\not\Leftrightarrow$ Markov process

Example: The simple symmetric random walk.

Let $(X_n, n \geq 1)$ be iid r.v.'s s.t. $\mathbb{P}(\{X_1 = +1\}) = \mathbb{P}(\{X_1 = -1\}) = \frac{1}{2}$

$S_0 = 0, \quad S_n = X_1 + \dots + X_n \quad n \geq 1$

$\mathcal{F}_0 = \{\phi, \Omega\}, \quad \mathcal{F}_n = \sigma(X_1, \dots, X_n) \quad n \geq 1$

Then $(S_n, n \in \mathbb{N})$ is a sq-int. martingale w.r.t $(\mathcal{F}_n, n \in \mathbb{N})$

Proof:

(i) $\mathbb{E}(S_n^2) = n \cdot \mathbb{E}(X_1^2) = n < +\infty \quad \forall n \geq 1$

(ii) S_n is $\mathcal{F}_n$-measurable: yes, as $S_n = X_1 + \dots + X_n$ ✓

(iii) $\mathbb{E}(S_{n+1} | \mathcal{F}_n) = \mathbb{E}(S_n + X_{n+1} | \mathcal{F}_n) = \mathbb{E}(S_n | \mathcal{F}_n) + \mathbb{E}(X_{n+1} | \mathcal{F}_n)$

$$= \underbrace{S_n}_{\mathcal{F}_n\text{-measurable}} + \underbrace{\mathbb{E}(X_{n+1})}_{X_{n+1} \perp\!\!\!\perp \mathcal{F}_n} = S_n \text{ a.s.}$$

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<u>Remark</u>: If $(M_n, n \in \mathbb{N})$ is a sq-int. martingale, then

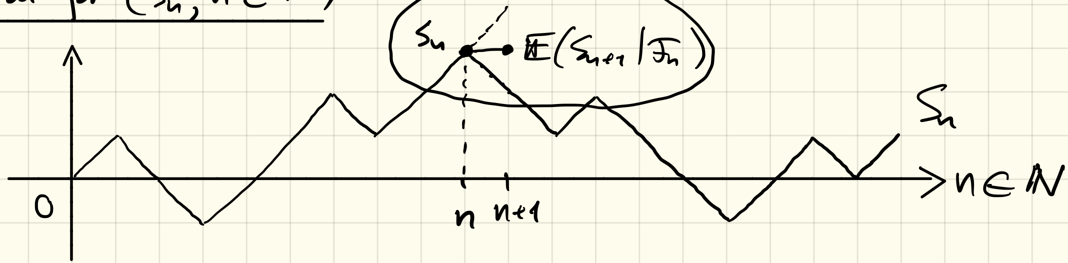
$$\mathbb{E}(M_{n+1}) = \mathbb{E}(\mathbb{E}(M_{n+1} | \mathcal{F}_n)) = \mathbb{E}(M_n) \quad \forall n \in \mathbb{N}$$

So $\mathbb{E}(M_n) = \mathbb{E}(M_{n-1}) = \ldots = \mathbb{E}(M_0) \quad \forall n \in \mathbb{N}$

So a martingale is a process with constant expectation; but it is also <u>much more</u> than that!

<u>Ex</u>: The process $(X_n, n \geq 1)$ with X_n iid r.v. has constant expectation, but it is not a martingale: $\mathbb{E}(X_{n+1} | \mathcal{F}_n) = \mathbb{E}(X_{n+1}) = 0$
$\neq X_n$

<u>Illustration for $(S_n, n \in \mathbb{N})$</u>:



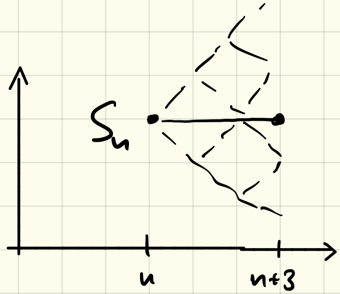
Property: Let $(M_n, n \geq 1)$ be a square-integrable martingale

- $\mathbb{E}(M_{n+2} \mid \mathcal{F}_n) = \mathbb{E}(\mathbb{E}(M_{n+2} \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n) = \mathbb{E}(M_{n+1} \mid \mathcal{F}_n)$

  $\mathcal{F}_n \subset \mathcal{F}_{n+1}$

  $= M_n$

- $\mathbb{E}(M_{n+m} \mid \mathcal{F}_n) = M_n \quad \forall n, m \in \mathbb{N}$



Remark: martingale $\neq$ Markov

- martingale property : $\mathbb{E}(M_{n+1} \mid \mathcal{F}_n) = M_n$

- Markov property : $\mathbb{P}(\{M_{n+1} \in B\} \mid \mathcal{F}_n) = \mathbb{P}(\{M_{n+1} \in B\} \mid M_n)$

  $\forall B \in \mathcal{B}(\mathbb{R})$

  equivalently :

  $\mathbb{E}(g(M_{n+1}) \mid \mathcal{F}_n) = \mathbb{E}(g(M_{n+1}) \mid M_n) \quad \forall g \in C_b(\mathbb{R})$

Sub- and supermartingales

(square-integrable)

✓

Def: A process $(M_n, n \in \mathbb{N})$ is a $\left.\begin{matrix}\text{sub-}\\ \text{super-}\end{matrix}\right\}$ martingale w.r.t. a filtration $(\mathcal{F}_n, n \in \mathbb{N})$ if:

(i) $\mathbb{E}(M_n^2) < +\infty \quad \forall n \in \mathbb{N}$

$\longrightarrow$ (ii) M_n is $\mathcal{F}_n$-measurable $\forall n \in \mathbb{N}$

(iii) $\mathbb{E}(M_{n+1} \mid \mathcal{F}_n) \substack{\geq \\ \leq} M_n$ a.s. $\forall n \in \mathbb{N}$

Rem: 1) if M is a $\left.\begin{matrix}\text{sub-}\\ \text{super-}\end{matrix}\right\}$ martingale, then $\mathbb{E}(M_{n+1}) \substack{\geq \\ \leq} \mathbb{E}(M_n)$.

2) Not every process is either a submartingale, or a martingale, or a supermartingale.

Example (simple asymmetric random walk)

$S_0 = 0, S_n = X_1 + \dots + X_n$ with $(X_n, n \geq 1)$ iid r.v.'s with

$$\mathbb{P}(\{X_1 = +1\}) = p = 1 - \mathbb{P}(\{X_1 = -1\}) \quad 0 < p < 1$$

$(S_n, n \in \mathbb{N})$ is a $\left.\begin{matrix}\text{sub-}\\ \text{super-}\end{matrix}\right\}$ martingale if $\left\{\begin{matrix} p \geq \frac{1}{2} \\ p \leq \frac{1}{2}\end{matrix}\right.$

Pf: $\mathbb{E}(S_{n+1} | \mathcal{F}_n) = \mathbb{E}(S_n | \mathcal{F}_n) + \mathbb{E}(X_{n+1} | \mathcal{F}_n) = S_n + \underbrace{\mathbb{E}(X_{n+1})}_{\left\{\begin{matrix}\geq 0 \\ \leq 0\end{matrix}\right.}$ ✓

Property: If M is a square-integrable martingale

& $\varphi : \mathbb{R} \to \mathbb{R}$ is a convex function st $\mathbb{E}(\varphi(M_n)^2) < +\infty$ $\forall n \in \mathbb{N}$

Then $(\varphi(M_n), n \in \mathbb{N})$ is a submartingale.

Pf: $\mathbb{E}(\varphi(M_{n+1}) | \mathcal{F}_n) \underset{\text{Jensen}}{\geq} \varphi(\mathbb{E}(M_{n+1} | \mathcal{F}_n)) = \varphi(M_n)$ ✓

Stopping times & optional stopping theorem

<u>Def</u>: A <u>random time</u> is a random variable $T: \Omega \to \mathbb{N}$

Let $(\mathcal{F}_n, n \in \mathbb{N})$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$.

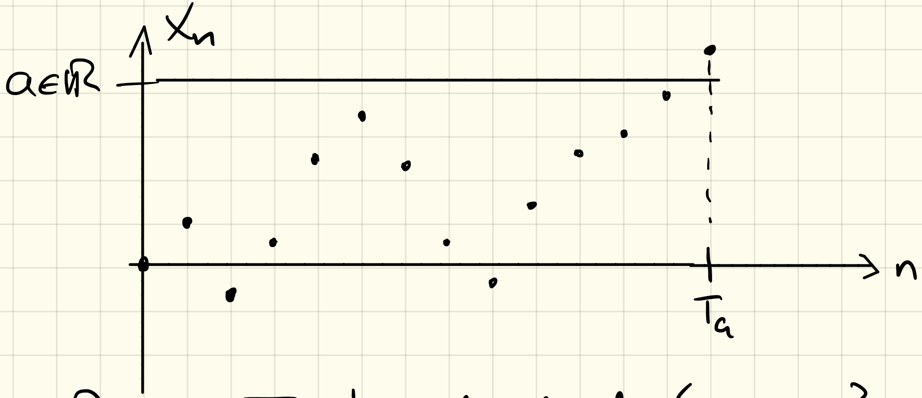
<u>Def</u>: A <u>stopping time</u> w.r.t. $(\mathcal{F}_n, n \in \mathbb{N})$ is a random time such that $\{\omega \in \Omega : T(\omega) = n\} \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$

<u>Ex</u>: Let $(X_n, n \in \mathbb{N})$ be a discrete-time stochastic process

$$\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n) \qquad n \in \mathbb{N}$$

Then $T_a = \inf\{n \in \mathbb{N} : X_n \geq a\}$ is a stopping time w.r.t. $(\mathcal{F}_n, n \in \mathbb{N})$

$a \in \mathbb{R}$



Proof: To be checked: $\{T_a = n\} \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$

$$\{T_a = n\} = \{X_k < a \ \forall 0 \le k \le n-1 \ \& \ X_n \ge a\}$$

$$= \Big(\bigcap_{0 \le k \le n-1} \underbrace{\{X_k < a\}}_{\in \mathcal{F}_k \subset \mathcal{F}_n} \Big) \cap \underbrace{\{X_n \ge a\}}_{\in \mathcal{F}_n} \in \mathcal{F}_n$$

✓

Def: If $(X_n, n \in \mathbb{N})$ is a discrete-time stochastic process and T is a random time, then we define

$$X_T(\omega) := X_{T(\omega)}(\omega) = \sum_{n \in \mathbb{N}} X_n(\omega) \, \mathbb{1}_{\{T=n\}}(\omega)$$

Def: We say that a random time is <u>bounded</u> if there exists $N \in \mathbb{N}$ such that $T(\omega) \leq N \quad \forall \omega \in \Omega$

$\left(\text{or } \mathbb{P}(\{T \leq N\}) = 1\right)$

Optional stopping theorem

- Let M be a (square-integrable) martingale w.r.t. a filtration $(\mathcal{F}_n, n \in \mathbb{N})$

  Let T be a stopping time w.r.t. $(\mathcal{F}_n, n \in \mathbb{N})$ such that $\exists N \in \mathbb{N}$ with $0 \le T(\omega) \le N \quad \forall \omega \in \Omega$

  Then $\mathbb{E}(M_0) = \mathbb{E}(M_T) = \mathbb{E}(M_N)$

- If M is a (square-integrable) submartingale, then $(\mathbb{E}(M_0) \le) \, \mathbb{E}(M_T) \le \mathbb{E}(M_N)$

  not proven here

Proof:

As $0 \le T(\omega) \le N$ $\forall \omega \in \Omega$, we have

$$M_T(\omega) = M_{T(\omega)}(\omega) = \sum_{n=0}^{N} M_n(\omega) \cdot 1_{\{T=n\}}(\omega)$$

$$\mathbb{E}(M_T) = \sum_{n=0}^{N} \mathbb{E}(M_n \cdot 1_{\{T=n\}}) = \sum_{n=0}^{N} \mathbb{E}\big(\mathbb{E}(M_N \mid \mathcal{F}_n) \cdot \overset{\in \mathcal{F}_n}{1_{\{T=n\}}}\big)$$

$\le$ (in case M is a submartingale)

$$= \sum_{n=0}^{N} \mathbb{E}\big(\mathbb{E}(M_N \cdot 1_{\{T=n\}} \mid \mathcal{F}_n)\big) = \sum_{n=0}^{N} \mathbb{E}(M_N \cdot 1_{\{T=n\}})$$

$$= \mathbb{E}\Big(M_N \cdot \underbrace{\sum_{n=0}^{N} 1_{\{T=n\}}}_{=1}\Big) = \mathbb{E}(M_N) \underset{(M = \text{martingale})}{=} \mathbb{E}(M_0)$$

$\#$

When does the optional stopping theorem fail?

- If T is not a stopping time:

Ex: T = value of $n \in \{0 .. N\}$ such that $M_n \geq M_m$ $\forall m \in \{0..N\}$

Then both $\mathbb{E}(M_T) > \mathbb{E}(M_0)$ & $\mathbb{E}(M_T) > \mathbb{E}(M_N)$

- If T is not bounded:

Ex: Consider $(X_n, n \geq 1)$ be a sequence of iid r.v.'s

s.t. $\mathbb{P}(\{X_1 = +1\}) = \mathbb{P}(\{X_1 = -1\}) = \frac{1}{2}$; $M_0 = 0$, $M_n = X_1 + \dots + X_n$, $n \geq 1$

M = martingale, $T_a = \inf\{n \in \mathbb{N} : M_n \geq a\}$ $a \in \mathbb{N}^*$

Then $\mathbb{E}(\underbrace{M_{T_a}}_{=a}) = a > 0 = \mathbb{E}(M_0)$.

Why is it an interesting theorem?

Consider again the previous example with M = simple symmetric random walk: $M_n = X_1 + \dots + X_n$.

X's = coin tosses $\begin{cases} \to +1 \text{ fr.} \\ \to -1 \text{ fr} \end{cases}$ $\Rightarrow$ M_n = cumulated gain at time n

$\forall N \geq 1, \quad \mathbb{E}(M_N) = \mathbb{E}(M_0) = 0$

Let $T = \min\{\inf\{n \in \mathbb{N} : M_n \geq 10\}, N\}$

$$\mathbb{E}(\underbrace{M_T}) = \mathbb{E}(M_0)$$

$\hookrightarrow \begin{cases} = +10 & \text{with high probability} \\ = -x & \text{with tiny probability (when } T = N) \end{cases}$