APA lecture 8

Extension of the weak law of large numbers : an example

Toss a fair coin until it falls on heads and call T the total number of tosses ; reward $= G = 2^T$ francs

Expected reward : $\mathbb{E}(G)$

$$\begin{cases} \mathbb{P}(\{G = 2^k\}) = \mathbb{P}(\{T = k\}) = \frac{1}{2^k} & \forall k \geq 1 \\ \mathbb{E}(G) = \sum_{k \geq 1} 2^k \, \mathbb{P}(\{G = 2^k\}) = \sum_{k \geq 1} \cancel{2^k} \cdot \frac{1}{\cancel{2^k}} = +\infty \end{cases}$$

St-Petersburg's paradox

SLLN :

After n games : reward $= G_1 + \ldots + G_n$
(iid)

$$\frac{G_1 + \ldots + G_n}{n} \xrightarrow[n \to \infty]{} +\infty \text{ a.s.}$$

$(S_n = G_1 + \dots + G_n)$

Proposition

Under the above assumptions, $\dfrac{S_n}{n \log_2 n} \xrightarrow[n\to\infty]{\mathbb{P}} 1$ (*)

So the amount you should bet per game is $\sim \log_2 n$ (in order to be allowed to play n games).

(*) means $\forall \varepsilon > 0$, $\mathbb{P}\left(\left\{\left|\dfrac{S_n}{n \log_2 n} - 1\right| \geq \varepsilon\right\}\right) \xrightarrow[n\to\infty]{} 0$

Proof: fix $n \geq 1$:

Define for $1 \leq j \leq n$: $\begin{cases} H_j = G_j \cdot 1_{G_j \leq n \log_2 n} \\ S_n' = H_1 + \dots + H_n \end{cases}$

$$\mathbb{P}\left(\left\{\left|\frac{S_n}{n\log_2 n}-1\right|\geq\varepsilon\right\}\right)=\mathbb{P}\left(\left\{|S_n-n\log_2 n|\geq\varepsilon n\log_2 n\right\}\right)$$

$$=\mathbb{P}\left(\left\{|S_n-n\log_2 n|\geq\varepsilon n\log_2 n,\ G_j\leq n\log_2 n\ \ \forall 1\leq j\leq n\right\}\right)$$

$$+\mathbb{P}\left(\left\{|S_n-n\log_2 n|\geq\varepsilon n\log_2 n,\ \exists 1\leq j\leq n\text{ s.t. } G_j\geq n\log_2 n\right\}\right)$$

$$\leq\mathbb{P}\left(\left\{|S_n'-n\log_2 n|\geq\varepsilon n\log_2 n,\ G_j\leq n\log_2 n\ \ \forall 1\leq j\leq n\right\}\right)$$

$$+\mathbb{P}\left(\left\{\exists 1\leq j\leq n\text{ s.t. } G_j\geq n\log_2 n\right\}\right)$$

$$\leq\underbrace{\mathbb{P}\left(\left\{|S_n'-n\log_2 n|\geq\varepsilon n\log_2 n\right\}\right)}_{①}+\underbrace{\mathbb{P}\left(\left\{\exists 1\leq j\leq n\text{ s.t. } G_j\geq n\log_2 n\right\}\right)}_{②}$$

$$② \leq\sum_{j=1}^{n}\mathbb{P}\left(\left\{G_j\geq n\log_2 n\right\}\right)=n\,\mathbb{P}\left(\left\{G_1\geq n\log_2 n\right\}\right)=n\sum_{k\geq\log_2(n\log_2 n)}\frac{1}{2^k}$$

$$\cong\frac{n}{2^{\log_2(n\log_2 n)}}=\frac{n}{n\log_2 n}=\frac{1}{\log_2 n}\xrightarrow[n\to\infty]{}0$$

$$(1) = \mathbb{P}\left(\left\{\left|S_n' - \underline{n\log_2 n}\right| \geq \varepsilon\, n\log_2 n\right\}\right) \qquad S_n' = \underline{H_1 + \dots + H_n}$$

$$\mathbb{E}(H_1) = \sum_{k \leq \log_2(n\log_2 n)} \cancel{2^k} \cdot \frac{1}{\cancel{2^k}} \cong \log_2(n\log_2 n) \cong \log_2 n$$

So $\mathbb{E}(S_n') = n\,\mathbb{E}(H_1) \cong n\log_2 n$

Chebyshev : $(1) \leq \dfrac{\mathbb{E}\left(\left(S_n' - \mathbb{E}(S_n')\right)^2\right)}{\varepsilon^2 n^2 (\log_2 n)^2} = \dfrac{\operatorname{Var}(S_n')}{\varepsilon^2 n^2 (\log_2 n)^2}$

$\psi(x) = x^2$

$$\operatorname{Var}(S_n') = n\operatorname{Var}(H_1) \leq n\,\mathbb{E}(H_1^2) = n\sum_{k \leq \log_2(n\log_2 n)} 2^{2k} \cdot \frac{1}{2^k}$$

$$= n \sum_{k \leq \log_2(n\log_2 n)} 2^k \cong n \cdot 2^{\log_2(n\log_2 n)}$$

$$= n \cdot n\log_2 n = n^2 \log_2 n$$

So $(1) \leq \dfrac{n^2\log_2 n}{\varepsilon^2 n^2 (\log_2 n)^2} = \dfrac{1}{\varepsilon^2 (\log_2 n)} \xrightarrow[n\to\infty]{} 0$ #

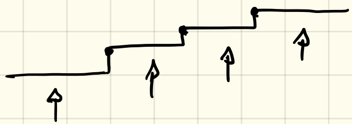
Convergence in distribution

Def: Let $(X_n, n \geq 1)$ be a sequence of random variables, not necessarily all defined on the same probability space. The sequence converges in distribution towards another limiting random variable X if

$$F_{X_n}(t) \xrightarrow[n\to\infty]{} F_X(t) \quad \forall t \in \mathbb{R} \text{ which is a continuity point of } F_X$$

equiv.: $\mathbb{P}(\{X_n \leq t\}) \xrightarrow[n\to\infty]{} \mathbb{P}(\{X \leq t\})$ 〃

Notation: $X_n \xrightarrow[n\to\infty]{d} X$



F_X

Ex: Let $(X_n, n \geq 1)$ be a sequence of iid r.v.

It is not the case that $X_n \xrightarrow{L^2} X$, nor that $X_n \xrightarrow{\mathbb{P}} X$, nor $X_n \longrightarrow X$ a.s. (except in the trivial case where $X_n \equiv c$)

But $F_{X_n}(t) = F_{X_1}(t) \; \forall n$! So the sequence $(X_n, n \geq 1)$ does converge in distribution!

Proposition: Let $(X_n, n \geq 1)$ be a sequence of r.v. defined on a common prob. space $(\Omega, \mathcal{F}, \mathbb{P})$ and let X be another r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$.

If $X_n \xrightarrow[n \to \infty]{\mathbb{P}} X$, then $X_n \xrightarrow[n \to \infty]{d} X$.

$$\left(\begin{tikzcd} L^2 \arrow[dr, Rightarrow] & & \text{a.s.} \arrow[dl, Rightarrow] \\ & \mathbb{P} \arrow[d, Rightarrow] & \\ & d & \end{tikzcd}\right)$$

Proof: $X_n \xrightarrow[n\to\infty]{\mathbb{P}} X$ means: $\forall \varepsilon>0$, $\mathbb{P}(\{|X_n-X|\geq\varepsilon\}) \xrightarrow[n\to\infty]{} 0$

To be shown: this implies that $F_{X_n}(t) \xrightarrow[n\to\infty]{} F_X(t)$

$\forall t\in\mathbb{R}$ continuity point of F_X.

- Let $\varepsilon>0$: $F_{X_n}(t) = \mathbb{P}(\{X_n\leq t\}) = \mathbb{P}(\{X_n\leq t,\ X\leq t+\varepsilon\})$

$$+\,\mathbb{P}(\{X_n\leq t,\ X>t+\varepsilon\}) \leq \underbrace{\mathbb{P}(\{X\leq t+\varepsilon\})}_{=F_X(t+\varepsilon)} + \underbrace{\mathbb{P}(\{|X_n-X|\geq\varepsilon\})}_{\xrightarrow[n\to\infty]{} 0}$$

So $\limsup\limits_{n\to\infty} F_{X_n}(t) \leq F_X(t+\varepsilon)$

- Let $\varepsilon>0$: $F_X(t-\varepsilon) = \mathbb{P}(\{X\leq t-\varepsilon\}) = \mathbb{P}(\{X\leq t-\varepsilon,\ X_n\leq t\})$

$$+\,\mathbb{P}(\{X\leq t-\varepsilon,\ X_n>t\}) \leq \underbrace{\mathbb{P}(\{X_n\leq t\})}_{=F_{X_n}(t)} + \underbrace{\mathbb{P}(\{|X_n-X|\geq\varepsilon\})}_{\xrightarrow[n\to\infty]{} 0}$$

So $\liminf\limits_{n\to\infty} F_{X_n}(t) \geq F_X(t-\varepsilon)$

So $F_X(t-\varepsilon) \le \liminf\limits_{n\to\infty} F_{X_n}(t) \le \limsup\limits_{n\to\infty} F_{X_n}(t) \le F_X(t+\varepsilon)$

$\forall \varepsilon > 0$

Letting $\varepsilon \to 0$: then if t is a continuity point of F_X,

we have $\lim\limits_{\varepsilon\to 0} F_X(t-\varepsilon) = \lim\limits_{\varepsilon\to 0} F_X(t+\varepsilon) = F_X(t)$

In this case, $\liminf\limits_{n\to\infty} F_{X_n}(t) = \limsup\limits_{n\to\infty} F_{X_n}(t) = \lim\limits_{n\to\infty} F_{X_n}(t) = F_X(t)$

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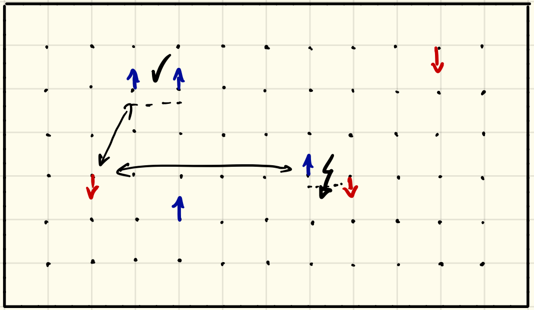
Application: the Curie-Weiss model

n spins $(n \sim 6.02 \cdot 10^{23})$



$X_j \in \{-1,+1\} \quad 1 \leq j \leq n$

magnetization: $M_n = \frac{1}{n}(X_1 + \ldots + X_n)$

Ising model: nearest neighbour interaction

β = inverse temperature > 0

Curie-Weiss model: mean field model: all spins interact!

$$\Omega_n = \{-1,+1\}^n, \quad \mathcal{F}_n = \mathcal{P}(\Omega_n), \quad \mathbb{P}_n(\{\omega\}) = \frac{1}{C_n} \cdot \exp\Big(\frac{\beta}{n} \underbrace{\sum_{\substack{j,k=1 \\ j<k}}^{n} \omega_j \cdot \omega_k}\Big)$$

$X_j(\omega) = \omega_j \quad 1 \leq j \leq n$

normalization constant

Proposition:

- If $\beta \le 1$ (high temperature), then $M_n \xrightarrow[n\to\infty]{d} 0$
- If $\beta > 1$ (low temperature), then $M_n \xrightarrow[n\to\infty]{d} M = \begin{cases} +m^*(\beta) & \text{wp } \frac{1}{2} \\ -m^*(\beta) & \text{wp } \frac{1}{2} \end{cases}$

Idea of the proof:

Aim: estimate $\mathbb{P}_n(\{M_n = m\})$ when n gets large

$$\mathbb{P}_n(\{\omega\}) = \frac{1}{C_n} \exp\Big(\frac{\beta}{n} \underbrace{\sum_{\substack{j,k=1 \\ j<k}}^{n} \omega_j \omega_k}_{} \Big)$$

$$= \frac{1}{2} \cdot \sum_{\substack{j,k=1 \\ j\neq k}}^{n} \omega_j \omega_k = \frac{1}{2}\Big(\sum_{j,k=1}^{n} \omega_j \omega_k - \underbrace{\sum_{j=1}^{n} \omega_j^2}_{=n} \Big)$$

$$= \frac{1}{\tilde{C}_n} \cdot \exp\Big(\frac{\beta}{2n} \sum_{j,k=1}^{n} \omega_j \omega_k \Big) = \frac{1}{\tilde{C}_n} \exp\Big(\frac{\beta}{2n} \Big(\underbrace{\sum_{j=1}^{n} \omega_j}_{= n \cdot M_n(\omega)} \Big)^2 \Big)$$

$$\mathbb{P}_n(\{\omega\}) = \frac{1}{\tilde{C}_n} \cdot \exp\left(+ \frac{\beta}{2n}\left(n \cdot M_n(\omega)\right)^2\right)$$

$$= \frac{1}{\tilde{C}_n} \cdot \exp\left(+ \frac{n\beta M_n(\omega)^2}{2}\right)$$

$$\mathbb{P}_n\left(\{\omega \in \Omega_n : M_n(\omega) = m\}\right) \qquad m \in \left\{-1, -1+\frac{2}{n}, \dots\dots, +1-\frac{2}{n}, +1\right\}$$

$$= \sum_{\omega \in \Omega_n : \frac{1}{n}(\omega_1 + \dots + \omega_n) = m} \mathbb{P}_n(\{\omega\}) = \frac{1}{\tilde{C}_n} \cdot \sum_{\underbrace{\omega \in \Omega_n : \frac{1}{n}(\omega_1 + \dots + \omega_n) = m}} \exp\left(-\frac{n\beta m^2}{2}\right)$$

$$= \frac{1}{\tilde{C}_n} \cdot \exp\left(+ \frac{n\beta m^2}{2}\right) \cdot \underbrace{\#\left\{\omega \in \Omega_n : \frac{1}{n}(\omega_1 + \dots + \omega_n) = m\right\}}$$

$$\underset{n \text{ large}}{\simeq} \exp\left(n \, h\left(\underbrace{\frac{1+m}{2}}_{\in [0,1]}\right)\right)$$

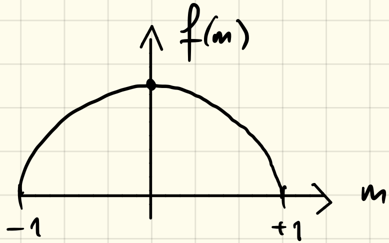
$$h(p) = -p \log p - (1-p)\log(1-p) \quad \text{entropy function}$$

So $$\mathbb{P}(\{M_n = m\}) \simeq \frac{1}{\tilde{C}_n} \cdot \exp\Big(n \underbrace{\Big(\frac{\beta m^2}{2} + h\Big(\frac{1+m}{2}\Big)\Big)}_{= f(m)}\Big)$$

As n gets large, the above probability will get even larger for values of m such that $f(m)$ is maximum

if $\beta \leq 1$:

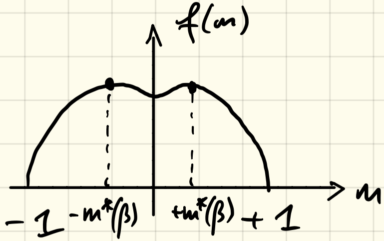
(high temperature)



$$M_n \xrightarrow[n\to\infty]{d} 0$$

if $\beta > 1$:

(low temperature)



$$M_n \xrightarrow[n\to\infty]{d} M$$

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