APA quiz 4

1. Which of the following functions $g : \mathbb{R} \to \mathbb{R}$ are bounded?
(guaranteeing therefore that $\mathbb{E}(g(X))$ is well defined for any random variable X)

a) $g(x) = x$

b) $g(x) = \max(\min(x, a), b)$ where $a, b \in \mathbb{R}$

c) $g(x) = \int_1^x \frac{1}{t}\, dt$ for $x \geq 1$ & $g(x) = 0$ for $x < 1$

d) $g(x) = \tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

e) $g(x) = \sum_{k \geq 1} \frac{1}{k} \mathbb{1}_{]k-1, k]}(x)$

f) $g(x) = \dfrac{1}{(x-1)^2}$

2. Which of the following random variables X have a finite expectation $\mathbb{E}(|X|) < +\infty$?

a) $\mathbb{P}(\{X=n\}) = \frac{C}{n+1} \quad n \geq 1$, where $C^{-1} = \sum_{n\geq 1} \frac{1}{n+1}$

b) $\mathbb{P}(\{X=n\}) = \frac{C}{n^2} \quad n \geq 1$, where $C^{-1} = \sum_{n\geq 1} \frac{1}{n^2}$

c) $\mathbb{P}(\{X=n\}) = 2^{-n} \quad n \geq 1$

d) X is continuous and $p_X(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \quad x \in \mathbb{R}$

e) X is continuous and $p_X(x) = \frac{1}{\pi(1+x^2)} \quad x \in \mathbb{R}$

f) $X = 1/U$, where U is a uniform random variable on $[0,1]$

3. Let X, Y be two square-integrable random variables. Which of the following statements is correct?

a) $\mathrm{Var}(X) \leq \mathbb{E}(X)^2$

b) If $X \geq 0$ and $Y \geq 0$, then $\mathrm{Cov}(X,Y) \geq 0$

c) If X, Y are uncorrelated, then $\mathbb{E}(X^2 Y^2) = \mathbb{E}(X^2)\,\mathbb{E}(Y^2)$

d) If X, Y are independent, then $\mathrm{Var}(XY) = \mathrm{Var}(X) \cdot \mathrm{Var}(Y)$

e) If X, Y are centered, then $\mathrm{Cov}(X,Y) = 0$

f) If $X \geq 0$ a.s. and $\mathrm{Cov}(X,Y) < 0$, then $Y \leq 0$ a.s.

4. Let X be a random variable and ϕ_X be its characteristic function. Which of the following statements are correct?

a) $t \to \phi_X(t)$ is a decreasing function on $\mathbb{R}_+$.

b) If $X \geqslant 0$ a.s., then $\phi_X(t) \geq 0 \quad \forall t \in \mathbb{R}_+$.

c) If $\int_{\mathbb{R}} |\phi_X(t)| \, dt < +\infty$, then X is a continuous r.v.

d) If $\phi_X(0) = 2$, then $\mathbb{P}(\{X \geqslant 0\}) = \frac{1}{2}$.

e) If X is bounded, then ϕ_X is differentiable.

f) If $\phi_{X+X}(t) = \phi_X(t) \cdot \phi_X(t) \quad \forall t \in \mathbb{R}$, then $X \perp\!\!\!\perp X$.