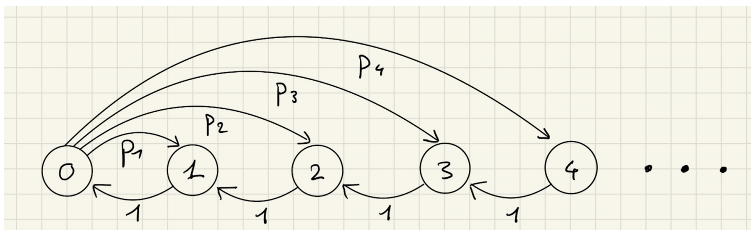
Final exam

SURNAME: .... FIRST NAME: ....

Exercise 1. (20 points) Let $(p_j,\, j \geq 1)$ be a sequence of non-negative numbers such that $\sum_{j\geq 1} p_j = 1$. Let also $(X_n,\, n \geq 0)$ be a time-homogeneous Markov chain with state space $S = \mathbb{N} = \{0, 1, 2, 3, \dots\}$ and transition matrix P represented by the following transition graph:



i.e., $p_{0j} = p_j$ and $p_{j,j-1} = 1$ for every $j \geq 1$ (and all other terms in the matrix P are equal to 0).

a) Under what minimal condition on the sequence $(p_j,\, j \geq 1)$ is the chain X irreducible?

b) Under what minimal condition on the sequence $(p_j,\, j \geq 1)$ is the chain X aperiodic?

Let us assume from now on that $p_j > 0$ for every $j \geq 1$. (*Hint for the above two questions*: Under this condition, the chain X is irreducible and aperiodic).

c) Show that under this assumption, the chain X is always recurrent.

Hint: Let $T_0 = \inf\{n \geq 1 \,:\, X_n = 0\}$ be the first return time to state 0.
Compute $f_{00}^{(n)} = \mathbb{P}(T_0 = n | X_0 = 0)$ for $n \geq 1$ and $f_{00} = \sum_{n\geq 1} f_{00}^{(n)}$.

d) Under what minimal condition on the sequence $(p_j,\, j \geq 1)$ is the chain X also positive-recurrent?

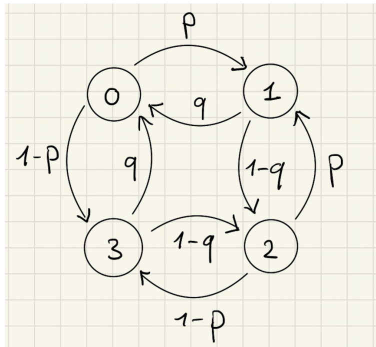
Hint: Compute $\mathbb{E}(T_0 | X_0 = 0)$.

e) Under the condition found in part d), compute the stationary distribution π of the chain X. Is it also a limiting distribution? Is detailed balance satisfied?

f) In the particular case where $p_j = 2^{-j}$ for $j \geq 1$, compute explicitly the stationary distribution π.

Exercise 2. (18+5 points)

Let $(X_n,\, n \geq 0)$ be a time-homogeneous Markov chain with state space $S = \{0, 1, 2, 3\}$ and transition matrix P represented by the following transition graph:



where $0 < p, q < 1$.

a) Compute the stationary distribution π of the chain. Is detailed balance satisfied for all parameters $0 < p, q < 1$?

b) Compute the eigenvalues $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3$ of the matrix P.

Hint: What is the rank of P?

c) Let $0 \leq \alpha \leq 1$ and $\widetilde{P} = \alpha\, I + (1 - \alpha)\, P$ be the transition matrix of the lazy Markov chain $(\widetilde{X}_n,\, n \geq 0)$.

c1) Compute the spectral γ of $\widetilde{P}$, as a function of α.

c2) For what value of α is γ maximal?

Let us assume from now on that α takes the value found in c2).

c3) Deduce an upper bound on $\|\widetilde{P}_0^n - \pi\|_{\mathrm{TV}}$.

BONUS d) Show that $P^3 = P$ and deduce by induction that we have the following equality for *n even*:

$$\widetilde{P}^n = \alpha^n\, I + \frac{1}{2}\,(1 - \alpha^n)\,(P + P^2) \tag{1}$$

e) Use equation (1) to compute the value of $\|\widetilde{P}_0^n - \pi\|_{\mathrm{TV}}$ for n even.

(Please pay attention that e) is NOT a bonus question!)

Exercise 3. (12 points)

Let $\beta_1, \beta_2 > 0$. On the set $S = \mathbb{Z}^2 = \{x = (x_1, x_2) \,:\, x_1 \in \mathbb{Z}, x_2 \in \mathbb{Z}\}$, one defines the distribution:

$$\pi_x = \frac{1}{Z} \exp(-\beta_1 \, x_1^2 - \beta_2 \, x_2^2) \quad \text{for } x = (x_1, x_2) \in \mathbb{Z}^2$$

where $Z = \sum_{x \in \mathbb{Z}^2} \exp(-\beta_1 \, x_1^2 - \beta_2 \, x_2^2)$.

Define now a base chain on S whose transition probabilities are given by

$$\psi_{xy} = \begin{cases} \frac{1}{4} & \text{if } y = x \pm e_1 \text{ or } y = x \pm e_2 \\ \\ 0 & \text{otherwise} \end{cases}$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The idea is then to use the Metropolis algorithm in order to sample from π.

a) Is this base chain irreducible? aperiodic? Does it hold that $\psi_{xy} > 0$ if and only if $\psi_{yx} > 0$?

b) Is this base chain ergodic?

For the rest of this exercise, assume in all your computations that $x_1 > 0$ and $x_2 > 0$.

c) Compute the acceptance probabilities a_{xy}, as well as the resulting transition probabilities p_{xy} of the Metropolis chain (not forgetting p_{xx}).

Hint: Simplify as much as possible the expression for a_{xy}: it will help you for the next questions !

d) If $\beta_1 < \beta_2$ and $x_1 = x_2$, is a_{xy} larger when $y = x + e_1$ or when $y = x + e_2$?

e) Is a_{xy} larger when $y = x + e_1$ and x_1 is small, or when $y = x + e_1$ and x_1 is large?

f) Describe the shape of the set of points x (in the quadrant $x_1 \geq 0$, $x_2 \geq 0$) where the acceptance probabilities are roughly equal for both $y = x + e_1$ and $y = x + e_2$.