Solutions to Homework 6

Exercise 1. a) For a given $\varepsilon > 0$, let us first consider n sufficiently large such that

$$\left| \frac{\mu_1 + \ldots + \mu_n}{n} - \mu \right| < \frac{\varepsilon}{2}$$

(such an n exists by assumption). For the same value of n, we have

$$\begin{aligned}
\mathbb{P}\left(\left\{\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right\}\right) &\leq \mathbb{P}\left(\left\{\left|\frac{S_n}{n} - \frac{\mu_1 + \ldots + \mu_n}{n}\right| \geq \frac{\varepsilon}{2}\right\}\right) \\
&= \mathbb{P}\left(\left\{\left|\sum_{i=1}^n (X_i - \mu_i)\right| \geq \frac{n\varepsilon}{2}\right\}\right) \leq \frac{4}{n^2\,\varepsilon^2}\,\mathbb{E}\left(\left(\sum_{i=1}^n (X_i - \mu_i)\right)^2\right) \\
&= \frac{4}{n^2\,\varepsilon^2} \sum_{i,j=1}^n \mathrm{Cov}(X_i, X_j) \leq \frac{4C_1}{n^2\,\varepsilon^2} \sum_{i,j=1}^n \exp(-C_2\,|i-j|) \\
&\leq \frac{8C_1}{n\,\varepsilon^2} \sum_{k \in \mathbb{Z}} \exp(-C_2\,|k|) \underset{n\to\infty}{\longrightarrow} 0, \quad \text{as} \quad \sum_{k\in\mathbb{Z}} \exp(-C_2\,|k|) < +\infty
\end{aligned}$$

b) All the steps of the proof made in class actually work. So yes, the strong law of large numbers also holds in this case.

c) We can check here that for $n \geq m \geq 1$, we have

$$\mathrm{Cov}(X_n, X_m) = a^{n-m}\,\mathrm{Var}(X_m)$$

and also that $\mathrm{Var}(X_1) = 0$ and

$$\mathrm{Var}(X_m) = 1 + a^2\,\mathrm{Var}(X_{m-1}) = \ldots = 1 + a^2 + a^4 + \ldots + a^{2(m-2)} \quad \text{for } m \geq 2$$

From this, we conclude that when $|a| < 1$, $\mathrm{Cov}(X_n, X_m)$ satisfies the condition given in the problem set. Besides, for every $n \geq 1$, we have

$$\mu_n = \mathbb{E}(X_n) = a\,\mathbb{E}(X_{n-1}) = a^{n-1}\,x$$

so

$$\lim_{n\to\infty} \frac{\mu_1 + \ldots + \mu_n}{n} = \frac{1}{n} \sum_{j=1}^n a^{j-1}\,x \underset{n\to\infty}{\to} 0$$

when $|a| < 1$, for any value of $x \in \mathbb{R}$. So $\mu = 0$ in this case and

$$\frac{S_n}{n} = \frac{X_1 + \ldots + X_n}{n} \underset{n\to\infty}{\to} 0 \quad \text{almost surely}$$

Exercise 2*. a) Because the random variables X_n are identically distributed and bounded, it holds that there exists $M > 0$ such that $|X_n(\omega)| \leq M$ for all $n \geq 1$ and $\omega \in \Omega$. (*Note:* all this could hold with probability 1 instead of $\forall \omega \in \Omega$). So it holds that

$$\frac{S_n}{n} - \frac{1}{n}\sum_{j=2}^{n} X_j = \frac{X_1}{n} \underset{n\to\infty}{\to} 0 \quad \text{almost surely}$$

Likewise, it holds for any $k \geq 1$ that

$$\frac{S_n}{n} - \frac{1}{n}\sum_{j=k}^{n} X_j \underset{n\to\infty}{\to} 0 \quad \text{almost surely}$$

meaning that

$$A = \left\{ \frac{S_n}{n} \text{ converges} \right\} = \left\{ \frac{1}{n}\sum_{j=k}^{n} X_j \text{ converges} \right\} \in \sigma(X_k, X_{k+1}, \ldots)$$

As this holds for every $k \geq 1$, this proves that $A \in \mathcal{T}$.

b) From the conclusion of Exercice 1.b), we know that $\mathbb{P}(A) = 1$ even when the X_n are just uncorrelated random variables, not necessarily independent. In order to find a counter-example, there remains therefore to find another event B.

To this end, let us consider the sequence $(Y_n,\, n \geq 1)$ of i.i.d. random variables such that $\mathbb{P}(\{Y_1 = +a\}) = 2/3$ and $\mathbb{P}(\{Y_1 = -2a\}) = 1/3$, and let Z be a random variable independent of the sequence $(Y_n,\, n \geq 1)$ such that $\mathbb{P}(\{Z = +1\}) = \mathbb{P}(\{Z = -1\}) = 1/2$. Let us finally define $X_n = Y_n \cdot Z$ for $n \geq 1$.

Choosing $a = 1/\sqrt{2}$, the random variables X_n have zero mean, unit variance and are uncorrelated, as

$$\mathbb{E}(X_n) = \mathbb{E}(Y_n) \cdot \mathbb{E}(Z) = 0, \quad \mathrm{Var}(X_n) = \mathbb{E}(X_n^2) = \mathbb{E}(Y_n^2) \cdot \mathbb{E}(Z^2) = \left(a^2\, \frac{2}{3} + 4a^2\, \frac{1}{3} \right) \cdot 1 = 2a^2 = 1$$

and for $n \neq m$:

$$\mathbb{E}(X_n \cdot X_m) = \mathbb{E}(Y_n \cdot Y_m) \cdot \mathbb{E}(Z^2) = \mathbb{E}(Y_n) \cdot \mathbb{E}(Y_m) \cdot 1 = (2a/3 - 2a/3)^2 = 0$$

Now, let $B = \{Z = +1\}$. The event B belongs to the tail σ-field $\mathcal{T}$ for the following reason: for any value of $n \geq 1$, the value of $X_n = Y_n \cdot Z$ determines the value of Z, as

$$Z = +1 \quad \text{if and only if} \quad X_n = +a \quad \text{or} \quad X_n = -2a$$

and likewise,

$$Z = -1 \quad \text{if and only if} \quad X_n = -a \quad \text{or} \quad X_n = +2a$$

So Z is measurable with respect to $\sigma(X_n) \subset \sigma(X_n, X_{n+1}, \ldots)$ for any $n \geq 1$, so Z is measurable with respect to $\mathcal{T} = \cap_{n\geq 1} \sigma(X_n, X_{n+1}, \ldots)$, i.e., $B = \{Z = +1\} \in \mathcal{T}$, but $\mathbb{P}(B) = 1/2 \notin \{0, 1\}$.