Solutions to Homework 13

Exercise 1. a) Let $(\mathcal{F}_n,\, n \geq 1)$ denote natural filtration of $(X_n, n \in \mathbb{N})$. Using the hint, we find

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = p\,(X_n^2 + 1) + (1-p)\,\frac{X_n}{2} \geq 2p\,X_n + (1-p)\,\frac{X_n}{2} = \frac{3p+1}{2}\,X_n \geq X_n$$

if $p \geq 1/3$. This condition turns out to be the minimal one. Indeed, it can always happen that X_n gets arbitrarily close to the value 1. In this case,

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = p\,(X_n^2 + 1) + (1-p)\,\frac{X_n}{2} \sim 2p + \frac{1-p}{2} = \frac{3p+1}{2}$$

which is strictly less than 1 if $p < \frac{1}{3}$.

b) When $p \geq \frac{1}{3}$, we have

$$\mathbb{E}(X_{n+1}) = \mathbb{E}(\mathbb{E}(X_{n+1}|\mathcal{F}_n)) \geq \mathbb{E}\left(\frac{3p+1}{2}\,X_n\right) = \frac{3p+1}{2}\,\mathbb{E}(X_n)$$

so

$$\mathbb{E}(X_n) \geq \left(\frac{3p+1}{2}\right)^n x$$

c) No. The justification for this is the following: it can always happen that X_n follows the "down" path so as to get arbitrarily close to the value zero. In this case,

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = p\,(X_n^2 + 1) + (1-p)\,\frac{X_n}{2} \sim p > X_n$$

so the supermartingale property (and therefore also the martingale property) fails to hold, for any fixed value of $p > 0$.

Exercise 2*. a) Observe first that for every value of $0 < p < 1$, we have $0 < M_{n+1} < 1$ whenever $0 < M_n < 1$ (and $M_0 = x \in]0,1[$ by assumption), so that it makes sense to consider both M_n and $1 - M_n$ as probabilities at every step. Let us then compute:

$$\mathbb{E}(M_{n+1}\,|\,\mathcal{F}_n) = pM_n\,(1-M_n) + ((1-p) + pM_n)\,M_n = M_n$$

for every $0 < p < 1$!

b) Again, let us compute:

$$\mathbb{E}(M_{n+1}\,(1-M_{n+1})\,|\,\mathcal{F}_n) = pM_n\,(1-pM_n)\,(1-M_n) + ((1-p)+pM_n)\,(1-(1-p)-pM_n)\,M_n$$
$$= pM_n\,(1-pM_n)\,(1-M_n) + ((1-p)+pM_n)\,p(1-M_n)\,M_n = p(2-p)\,M_n\,(1-M_n)$$

c) Therefore, we obtain by induction:

$$\mathbb{E}(M_n\,(1-M_n)) = (p(2-p))^n\,x(1-x)$$

which converges to 0 as n gets large (as $p(2-p) < 1$ for every $0 < p < 1$).

d) Because the martingale M is bounded, the answer is yes to all three questions.

e) The answer obtained in c) suggests that M_n converges either to 0 or 1 as $n \to \infty$, which turns out to be the case. Moreover, as seen in class:

$$\mathbb{P}(\{M_\infty = 1\}) = \mathbb{E}(M_\infty) = \mathbb{E}(M_0) = x, \quad \text{so} \quad \mathbb{P}(\{M_\infty = 0\}) = 1 - x$$

Exercise 3. a) Let us compute the function $\psi(m) = \mathbb{E}(|m+U|) = \frac{1}{2}\int_{-1}^{1} |m+u|\, du$. If $m \geq 1$, then

$$\psi(m) = \mathbb{E}(m+U) = m$$

while if $0 \leq m < 1$, then

$$\psi(m) = \frac{1}{2}\left(\int_{-1}^{-m} (-m-u)\, du + \int_{-m}^{+1} (m+u)\, du\right) = \frac{1+m^2}{2}$$

b) By what was computed in part a), we see that for $m \geq 0$, $\psi(m) \geq m$, so

$$\mathbb{E}(M_{n+1} \,|\, \mathcal{F}_n) = \mathbb{E}(|M_n + U_{n+1}| \,|\, \mathcal{F}_n) = \psi(M_n) \geq M_n$$

c)

$$A_{n+1} - A_n = \mathbb{E}(M_{n+1} - M_n | \mathcal{F}_n) = \psi(M_n) - M_n = \begin{cases} 0 & \text{if } M_n \geq 1 \\ \frac{1+M_n^2}{2} - M_n = \frac{(1-M_n)^2}{2} & \text{if } M_n < 1 \end{cases}$$

so $A_n = \frac{1}{2}\sum_{j=1}^{n-1}(1-M_j)^2\, 1_{\{M_j<1\}}$.

d) Yes:

$$\mathbb{E}(M_{n+1}^2 \,|\, \mathcal{F}_n) = \mathbb{E}((M_n + U_{n+1})^2 | \mathcal{F}_n) = M_n^2 + 2M_n \mathbb{E}(U_{n+1}) + \mathbb{E}(U_{n+1}^2) = M_n^2 + \frac{1}{3} \geq M_n^2$$

e) By the previous computation, we deduce that $c = \frac{1}{3}$.

f) No. M is a positive submartingale behaving like a random walk with uniform increments on the positive axis, and being reflected when it gets close to 0. It will not converge anywhere.