Homework 14

Exercise 1.

Part I. Let $(X_n,\, n \geq 1)$ be a sequence of i.i.d. random variables such that $\mathbb{P}(\{X_n = +1\}) = p$ and $\mathbb{P}(\{X_n = -1\}) = 1 - p$ for some fixed $0 < p < 1/2$.

Let $S_0 = 0$ and $S_n = X_1 + \ldots + X_n$, $n \geq 1$. Let also $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, $n \geq 1$.

Preliminary question. Deduce from Hoeffding's inequality that for any $0 < p < 1/2$,

$$\mathbb{P}(\{|S_n - n\,(2p-1)| \geq nt\}) \leq 2 \exp\left(-\frac{nt^2}{2}\right) \quad \forall t > 0,\, n \geq 1.$$

This inequality will be useful at some point in this exercise.

Let now $(Y_n,\, n \in \mathbb{N})$ be the process defined as $Y_n = \lambda^{S_n}$ for some $\lambda > 0$ and $n \in \mathbb{N}$.

a) *Using Jensen's inequality only*, for what values of λ can you conclude that the process Y is a submartingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$?

b) Identify now the values of $\lambda > 0$ for which it holds that the process $(Y_n = \lambda^{S_n},\, n \in \mathbb{N})$ is a martingale / submartingale / supermartingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

c) Compute $\mathbb{E}(|Y_n|)$ and $\mathbb{E}(Y_n^2)$ for every $n \in \mathbb{N}$ (and every $\lambda > 0$).

d) For what values of $\lambda > 0$ does it hold that $\sup_{n\in\mathbb{N}} \mathbb{E}(|Y_n|) < +\infty$? $\sup_{n\in\mathbb{N}} \mathbb{E}(Y_n^2) < +\infty$?

e) Run the process Y numerically. For what values of $\lambda > 0$ do you observe that there exists a random variable Y_∞ such that $Y_n \underset{n\to\infty}{\to} Y_\infty$ a.s.?

Prove it then theoretically and compute the random variable Y_∞ when it exists (this computation might depend on λ, of course).

f) For what values of $\lambda > 0$ does it hold that $Y_n \underset{n\to\infty}{\overset{L^2}{\to}} Y_\infty$?

g) Finally, for what values of $\lambda > 0$ does it hold that $\mathbb{E}(Y_\infty | \mathcal{F}_n) = Y_n$, $\forall n \in \mathbb{N}$?

Part II. Consider now the (interesting) value λ for which the process Y is a martingale. (Spoiler: there is a unique such value of λ, and it is greater than 1.)

Let $a \geq 1$ be an integer and consider the stopping time $T_a = \inf\{n \in \mathbb{N} : Y_n \geq \lambda^a \quad \text{or} \quad Y_n \leq \lambda^{-a}\}$.

a) Estimate numerically $\mathbb{P}(\{Y_{T_a} = \lambda^a\})$ for some values of a. Explain your method.

b) Is it true that $\mathbb{E}(Y_{T_a}) = \mathbb{E}(Y_0)$? Justify your answer.

c) If possible, use the previous statement to compute $P = \mathbb{P}(\{Y_{T_a} = \lambda^a\})$ theoretically. How fast does this probability decay with a?

Consider finally the other stopping time $T'_a = \inf\{n \in \mathbb{N} : Y_n \geq \lambda^a\}$.

d) Estimate numerically $\mathbb{P}(\{Y_{T'_a} = \lambda^a\})$ for some values of a. Explain your method.

e) Is it true that $\mathbb{E}(Y_{T'_a}) = \mathbb{E}(Y_0)$? Justify your answer.

f) If possible, use the above statement to compute $P' = \mathbb{P}(\{Y_{T'_a} = \lambda^a\})$ theoretically. Is this probability P' greater or smaller than P?