Homework 11

Exercise 1*. Let $(M_n,\, n \in \mathbb{N})$ be a submartingale with respect to a filtration $(\mathcal{F}_n,\, n \in \mathbb{N})$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be a Borel-measurable and convex function such that $\mathbb{E}(|\varphi(M_n)|) < +\infty$, $\forall n \in \mathbb{N}$.

a) What additional property of φ ensures that the process $(\varphi(M_n),\, n \in \mathbb{N})$ is also a submartingale?

b) In particular, which of the following two processes is ensured to be a submartingale: $(M_n^2,\, n \in \mathbb{N})$ and/or $(\exp(M_n),\, n \in \mathbb{N})$?

Let $(X_n,\, n \geq 1)$ be a sequence of i.i.d. random variables such $\mathbb{P}(\{X_1 = +1\}) = \mathbb{P}(\{X_1 = -1\}) = \frac{1}{2}$; let $S_0 = 0$ and $S_n = X_1 + \ldots + X_n$ for $n \geq 1$; finally, let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ for $n \geq 1$.

c) For which value of $c > 0$ is the process $(S_n^2 - cn,\, n \in \mathbb{N})$ is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$?

d) For which value of $c > 0$ is the process $\left(\frac{\exp(S_n)}{c^n},\, n \in \mathbb{N}\right)$ a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$?

Assume now that $\mathbb{P}(\{X_1 = +1\}) = p = 1 - \mathbb{P}(\{X_1 = -1\})$ for some $0 < p < 1$ with $p \neq \frac{1}{2}$.

e) Does there exist a number $c > 0$ such that the process $(S_n^2 - cn,\, n \in \mathbb{N})$ is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$? If yes, compute the value of c; otherwise, justify why it is not the case.

f) Does there exist a number $c > 0$ such that the process $\left(\frac{\exp(S_n)}{c^n},\, n \in \mathbb{N}\right)$ is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$? If yes, compute the value of c; otherwise, justify why it is not the case.

Exercise 2. a) Let $(M_n,\, n \in \mathbb{N})$ be an *increasing* martingale, that is, $M_{n+1} \geq M_n$ a.s. for all $n \in \mathbb{N}$. Show that $M_n = M_0$ a.s., for all $n \in \mathbb{N}$.

b) Let $(M_n,\, n \in \mathbb{N})$ be a square-integrable martingale such that $(M_n^2,\, n \in \mathbb{N})$ is also a martingale. Show that $M_n = M_0$ a.s., for all $n \in \mathbb{N}$.

Exercise 3. Let $(S_n,\, n \in \mathbb{N})$ be the simple symmetric random walk, $(\mathcal{F}_n,\, n \in \mathbb{N})$ be its natural filtration and

$$T = \inf\{n \geq 1 \,:\, S_n \geq a \quad \text{or} \quad S_n \leq -b\},$$

where a, b are positive integers.

a) Show that T is a stopping time with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

b) Use the optional stopping theorem to compute $\mathbb{P}(\{S_T = a\})$.

Let now $(M_n,\, n \in \mathbb{N})$ be defined as $M_n = S_n^2 - n$, for all $n \in \mathbb{N}$.

c) Show that the process $(M_n,\, n \in \mathbb{N})$ is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

d) Apply the optional stopping theorem to compute $\mathbb{E}(T)$.

Remark: Even though T is an unbounded stopping time, the optional stopping theorem applies both in parts b) and d). Notice that the theorem would *not* apply if one would consider the stopping time: $T' = \inf\{n \geq 1 \,:\, S_n \geq a\}$.