Solutions to Homework 11

Exercise 1*. a) By Jensen's inequality, we know that

$$\mathbb{E}(\varphi(M_{n+1})|\mathcal{F}_n) \geq \varphi(\mathbb{E}(M_{n+1}|\mathcal{F}_n))$$

We also know that $\mathbb{E}(M_{n+1}|\mathcal{F}_n) \geq X_n$, since $(M_n, n \in \mathbb{N})$ is a submartingale. The function φ needs therefore to be also increasing in order to ensure that

$$\mathbb{E}(\varphi(M_{n+1})|\mathcal{F}_n) \geq \varphi(M_n)$$

b) In particular, $(M_n^2,\, n \geq 1)$ need not be a submartingale, whereas $(\exp(M_n),\, n \geq 1)$ always is.

c) For every $n \in \mathbb{N}$, S_n^2 is bounded and $\mathcal{F}_n$-measurable, and

$$\begin{aligned}\mathbb{E}(S_{n+1}^2 \,|\, \mathcal{F}_n) &= \mathbb{E}(S_n^2 + 2S_n X_{n+1} + X_{n+1}^2 \,|\mathcal{F}_n) \\ &= S_n^2 + 2S_n\mathbb{E}(X_{n+1}) + \mathbb{E}(X_{n+1}^2) = S_n^2 + 1\end{aligned}$$

so we should take $c = 1$.

d) For every $n \in \mathbb{N}$, $\exp(S_n)/c^n$ is bounded and $\mathcal{F}_n$-measurable. Let us then compute

$$\begin{aligned}\mathbb{E}(\exp(S_{n+1}) \,|\, \mathcal{F}_n) &= \mathbb{E}(\exp(S_n + X_{n+1}) \,|\, \mathcal{F}_n) = \exp(S_n)\, \mathbb{E}(\exp(X_{n+1})) \\ &= \exp(S_n)\, \frac{e^{+1} + e^{-1}}{2} = \exp(S_n)\, \cosh(1)\end{aligned}$$

so we should take $c = \cosh(1)$.

e) The answer is no. Indeed, we have:

$$\mathbb{E}(S_{n+1}^2 \,|\, \mathcal{F}_n) = \mathbb{E}((S_n + X_{n+1})^2 \,|\, \mathcal{F}_n) = S_n^2 + 2S_n\, \mathbb{E}(X_{n+1}) + \mathbb{E}(X_{n+1}^2) = S_n^2 + 2(2p-1)\, S_n + 1$$

and there is no way to make the middle term disappear by considering $S_n^2 - cn$ with $c > 0$ instead of S_n^2.

f) In this case, the answer is yes:

$$\begin{aligned}\mathbb{E}(\exp(S_{n+1}) \,|\, \mathcal{F}_n) &= \mathbb{E}(\exp(S_n + X_{n+1}) \,|\, \mathcal{F}_n) = \exp(S_n)\, \mathbb{E}(\exp(X_{n+1})) \\ &= \exp(S_n)\, (p\, e^{+1} + (1-p)\, e^{-1})\end{aligned}$$

Taking therefore $c = p\, e^{+1} + (1-p)\, e^{-1}$, we obtain what we want.

Exercise 2. a) We know that $M_{n+1} - M_n \geq 0$ a.s., for all $n \geq 0$, and since M is a martingale, we also know that $\mathbb{E}(M_{n+1} - M_n) = 0$ for all $n \geq 0$, so $M_{n+1} = M_n$ a.s. for all $n \geq 0$, i.e. $M_n = M_0$ a.s. for all $n \geq 0$.

b) Let us compute, for $n \geq 0$,

$$\begin{aligned}\mathbb{E}((M_{n+1} - M_n)^2) &= \mathbb{E}(M_{n+1}^2 - 2M_{n+1}M_n + M_n^2) = \mathbb{E}(\mathbb{E}(M_{n+1}^2 - 2M_{n+1}M_n + M_n^2|\mathcal{F}_n)) \\ &= \mathbb{E}(\mathbb{E}(M_{n+1}^2|\mathcal{F}_n) - 2\mathbb{E}(M_{n+1}|\mathcal{F}_n)M_n + M_n^2) = \mathbb{E}(M_n^2 - 2M_n^2 + M_n^2) = 0\end{aligned}$$

where we have used the assumption that $\mathbb{E}(M_{n+1}^2|\mathcal{F}_n) = M_n^2$. Therefore, $M_n = M_0$ a.s. for all $n \geq 0$.

Exercise 3. a) For every $n \geq 0$, it holds that

$$\{T = n\} = \left(\cap_{j=0}^{n-1}\{-b < S_j < a\}\right) \bigcap \left(\{S_n \leq -b\} \cup \{S_n \geq a\}\right)$$

and as all these events belong to $\mathcal{F}_n$, so does $\{T = n\}$.

b) As a and b are positive integers, S_T can only take values in $\{-b, a\}$, so we obtain by the optional stopping theorem:

$$0 = \mathbb{E}(S_0) = \mathbb{E}(S_T) = a\,\mathbb{P}(S_T = a) - b\,\mathbb{P}(S_T = -b) = a\,\mathbb{P}(S_T = a) - b\,(1 - \mathbb{P}(\{S_T = a\}))$$

which implies that $\mathbb{P}(\{S_T = a\}) = \frac{b}{a+b}$ [quite a “logical” result, if you think about it].

c) For $n \geq 0$, let us compute

$$\begin{aligned}\mathbb{E}(M_{n+1}|\mathcal{F}_n) &= \mathbb{E}(S_{n+1}^2 - (n+1)|\mathcal{F}_n) = \mathbb{E}(S_n^2 + 2S_nX_{n+1} + X_{n+1}^2|\mathcal{F}_n) - (n+1) \\ &= S_n^2 + 2S_n\,\mathbb{E}(X_{n+1}) + \mathbb{E}(X_{n+1}^2) - (n+1) = S_n^2 + 0 + 1 - (n+1) = S_n^2 - n = M_n\end{aligned}$$

which proves the claim.

d) Applying again the optional stopping theorem, we obtain

$$0 = \mathbb{E}(M_0) = \mathbb{E}(M_T) = \mathbb{E}(S_T^2 - T) = \mathbb{E}(S_T^2) - \mathbb{E}(T)$$

So by part b), we obtain

$$\mathbb{E}(T) = \mathbb{E}(S_T^2) = a^2\,\mathbb{P}(\{S_T = a\}) + b^2\,\mathbb{P}(\{S_T = -b\}) = a^2\,\frac{b}{a+b} + b^2\,\frac{a}{a+b} = ab$$