Homework 12

Exercise 1. (If one cannot win on a game, then it is a martingale)
Let $(\mathcal{F}_n,\, n \in \mathbb{N})$ be a filtration and $(M_n,\, n \in \mathbb{N})$ be a process adapted to $(\mathcal{F}_n,\, n \in \mathbb{N})$ such that $\mathbb{E}(|M_n|) < \infty$, for all $n \in \mathbb{N}$.

Show that if for any predictable process $(H_n,\, n \in \mathbb{N})$ such that H_n is a bounded random variable $\forall n \in \mathbb{N}$, we have

$$\mathbb{E}((H \cdot M)_N) = 0, \quad \forall N \in \mathbb{N},$$

then $(M_n,\, n \in \mathbb{N})$ is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

Exercise 2. Let $(X_n,\, n \geq 1)$ be a family of independent square-integrable random variables such that $\mathbb{E}(X_n) = 0$ for all $n \geq 1$. Let $M_0 = 0$, $M_n = X_1 + \ldots + X_n$, $n \geq 1$.

The process $(M_n,\, n \in \mathbb{N})$ is a martingale, but it is also a process with independent increments. Show that $(M_n^2 - \mathbb{E}(M_n^2),\, n \in \mathbb{N})$ is also a martingale (hence the process A in the Doob decomposition of the submartingale $(M_n^2,\, n \in \mathbb{N})$ is a deterministic process in this case).

Exercise 3*. Let $(X_n,\, n \geq 1)$ be a sequence of i.i.d. random variables such that $\mathbb{P}(\{X_1 = +1\}) = \mathbb{P}(\{X_1 = -1\}) = \frac{1}{2}$. Let also $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ for $n \geq 1$ and let $(H_n,\, n \in \mathbb{N})$ be a predictable process with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\exists K_n > 0$ with $|H_n(\omega)| \leq K_n$ for all $\omega \in \Omega$. Let finally

$$G_0 = 0 \quad \text{and} \quad G_n = \sum_{j=1}^{n} H_j X_j, \quad n \geq 1.$$

From the course, we know that the process G is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

a) Under the assumptions made, is it possible that $\mathbb{E}(H_j X_j) > 0$ for some j? Explain!

b) Find the unique predictable and increasing process $(A_n,\, n \in \mathbb{N})$ such that the process $(G_n^2 - A_n,\, n \in \mathbb{N})$ is also a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

From now on, consider the particular case where $H_n(\omega) \in \{-1, +1\}$ for every $n \in \mathbb{N}$ and $\omega \in \Omega$.

c) Compute the process A in this particular case.

d) Let $a \geq 1$ be an integer and let $T = \inf\{n \geq 1 : |G_n| \geq a\}$. Compute $\mathbb{E}(T)$ [no full justification required here].