**Solutions to Homework 12**

**Exercise 1.** Let $m \in \mathbb{N}$ and $U$ be an $\mathcal{F}_m$-measurable and bounded random variable. Let us also define

$$H_n = \begin{cases} U, & \text{if } n = m+1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(H_n,\, n \in \mathbb{N})$ is predictable and for $m < N$, we have by assumption that $M_m$ is $\mathcal{F}_m$-measurable and also that

$$0 = \mathbb{E}((H \cdot M)_N) = \mathbb{E}(U(M_{m+1} - M_m)).$$

Therefore, $M_m = \mathbb{E}(M_{m+1}|\mathcal{F}_m)$, so $(M_n,\, n \in \mathbb{N})$ is a martingale.

**Exercise 2.** For all $n \geq 0$, we have

$$\mathbb{E}(M_{n+1}^2 - \mathbb{E}(M_{n+1}^2)|\mathcal{F}_n) = \mathbb{E}((M_n + X_{n+1})^2|\mathcal{F}_n) - \mathbb{E}((M_n + X_{n+1})^2)$$
$$= M_n^2 - 2M_n\,\mathbb{E}(X_{n+1}) + \mathbb{E}(X_{n+1}^2) - \mathbb{E}(M_n^2) - 2\mathbb{E}(M_n X_{n+1}) - \mathbb{E}(X_{n+1}^2) = M_n^2 - \mathbb{E}(M_n^2)$$

as $\mathbb{E}(M_n X_{n+1}) = \mathbb{E}(M_n)\,\mathbb{E}(X_{n+1}) = 0$.

**Exercise 3*.** a) No. $H_j$ is $\mathcal{F}_{j-1}$-measurable, while $X_j$ is independent of $\mathcal{F}_j$, so

$$\mathbb{E}(H_j X_j) = \mathbb{E}(H_j)\,\mathbb{E}(X_j) = \mathbb{E}(H_j)\,0 = 0$$

b) We have

$$A_{n+1} - A_n = \mathbb{E}(G_{n+1}^2|\mathcal{F}_n) - G_n^2 = \mathbb{E}((G_n + H_{n+1}X_{n+1})^2|\mathcal{F}_n) - G_n^2$$
$$= G_n^2 + 2G_n H_{n+1}\mathbb{E}(X_{n+1}) + H_{n+1}^2\mathbb{E}(X_{n+1}^2) - G_n^2 = H_{n+1}^2$$

so $A_0 = 0$ and $A_n = \sum_{j=1}^n H_j^2$.

c) $H_j^2 = 1$ for all $j$, so $A_n = n$.

d) Here, the idea is to use the optional stopping theorem with the martingale $(G_n^2 - n,\, n \in \mathbb{N})$, which gives

$$\mathbb{E}(G_T^2 - T) = \mathbb{E}(G_0^2 - 0) = 0, \quad \text{so} \quad \mathbb{E}(T) = \mathbb{E}(G_T^2) = a^2$$

Unfortunately, a full justification of the use of the theorem is impossible here using only the tools that you have learned in class, because the martingale $(G_n^2 - n,\, n \in \mathbb{N})$ is not bounded from below until the (unbounded) time $T$.