Homework 10

Exercise 1. Let $(X_n,\, n \geq 1)$ be a sequence of i.i.d. $\mathcal{E}(\lambda)$ random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., X_1 admits the following pdf:

$$p_{X_1}(x) = \begin{cases} \lambda \exp(-\lambda x) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Let also $S_n = X_1 + \ldots + X_n$. Using the large deviations principle, find a tight upper bound on

$$\mathbb{P}\left(\{S_n \geq nt\}\right) \quad \text{for } t > \mathbb{E}(X_1) = \frac{1}{\lambda}$$

Exercise 2*. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X be an integrable random variable defined on this space and let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Relying only on the definition of conditional expectation, show the following properties:

a) $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$.

b) If X is independent of $\mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ a.s.

c) If X is $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$ a.s.

d) If Y is $\mathcal{G}$-measurable and bounded, then $\mathbb{E}(XY|\mathcal{G}) = \mathbb{E}(X|\mathcal{G})\, Y$ a.s.

e) If $\mathcal{H}$ is a sub-σ-field of $\mathcal{G}$, then $\mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G}) = \mathbb{E}(X|\mathcal{H}) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H})$ a.s.

Hint for parts b) to e): According to the course definition, in order to check that some candidate random variable Z is the conditional expectation of X given $\mathcal{G}$, you should check the following two conditions:

(i) Z is $\mathcal{G}$-measurable;

(ii) Z satisfies $\mathbb{E}((Z - X)\, U) = 0$ for every U $\mathcal{G}$-measurable and bounded.

Exercise 3. Let X, Y be two discrete random variables (with values in a countable set C). Let us moreover assume that X is integrable.

a) Show that the random variable $\psi(Y)$, where ψ is defined as

$$\psi(y) = \sum_{x \in C} x\, \mathbb{P}(\{X = x\}|\{Y = y\})$$

matches the definition of conditional expectation $\mathbb{E}(X|Y)$ given in the lectures.

b) *Application:* One rolls two independent and balanced dice (say Y and Z), each with four faces. What is the conditional expectation of the maximum of the two, given the value of one of them?

Exercise 4. Let X be a random variable such that $\mathbb{P}(\{X = +1\}) = \mathbb{P}(\{X = -1\}) = \frac{1}{2}$ and $Z \sim \mathcal{N}(0,1)$ be independent of X. Let also $a > 0$ and $Y = aX + Z$. We propose below four possible estimators of the variable X given the noisy observation Y:

$$\widehat{X}_1 = \frac{Y}{a} \qquad \widehat{X}_2 = \frac{aY}{a^2+1} \qquad \widehat{X}_3 = \operatorname{sign}(aY) \qquad \widehat{X}_4 = \tanh(aY)$$

a) Which estimator among these four minimizes the mean square error (MSE) $\mathbb{E}((\widehat{X} - X)^2)$?

In order to answer the question, draw on the same graph the four curves representing the MSE as a function of $a > 0$. For this, you may use either the exact mathematical expression of the MSE or the one obtained via Monte-Carlo simulations.

b) Provide a theoretical justification for your conclusion.

c) For which of the four estimators above does it hold that $\mathbb{E}((\widehat{X} - X)^2) = \mathbb{E}(X^2) - \mathbb{E}(\widehat{X}^2)$?