**Solutions to Homework 8**

**Exercise 1.** a) Let us compute first

$$\mathbb{E}(S_1) = \frac{1}{2}\left(\frac{3S_0}{2} + \frac{S_0}{2}\right) = S_0$$

Assuming now that $\mathbb{E}(S_n) = S_0$ (more precisely, that the expectation stays constant over $n$ coin tosses), let us compute $\mathbb{E}(S_{n+1})$:

$$\begin{aligned}\mathbb{E}(S_{n+1}) &= \mathbb{E}(S_{n+1}|\{X_1 = +1\})\,\mathbb{P}(\{X_1 = +1\}) + \mathbb{E}(S_{n+1}|\{X_1 = -1\})\,\mathbb{P}(\{X_1 = -1\}) \\ &= \frac{1}{2}\left(\mathbb{E}(S_{n+1}|\{S_1 = \frac{3S_0}{2}\}) + \mathbb{E}(S_{n+1}|\{S_1 = \frac{S_0}{2}\})\right) = \frac{1}{2}\left(\frac{3S_0}{2} + \frac{S_0}{2}\right) = S_0\end{aligned}$$

*Note:* The computation is slightly unorthodox here, but we will see a cleaner way to prove this later in the course.

b) $Y_n$ is the sum of $n$ i.i.d. random variables, as the following computation shows:

$$Y_n = \log\left(\frac{S_n}{S_0}\right) = \log\left(\prod_{j=1}^n \left(1 + \frac{X_j}{2}\right)\right) = \sum_{j=1}^n \log\left(1 + \frac{X_j}{2}\right)$$

and these random variables are bounded, so by the central limit theorem,

$$\frac{Y_n - n\,\mu}{\sqrt{n}\,\sigma} \underset{n\to\infty}{\overset{d}{\to}} Z \sim \mathcal{N}(0,1)$$

where $\mu = \mathbb{E}(\log(1 + X_1/2)) = \frac{1}{2}(\log(3/2) + \log(1/2)) \simeq -0.144$ and

$$\sigma^2 = \mathrm{Var}(\log(1 + X_1/2)) = \frac{1}{2}(\log(3/2)^2 + \log(1/2)^2) - \mu^2 \simeq 0.3$$

This is saying that for large $n$, we have

$$Y_n \simeq -0.144n + \sqrt{0.26n}\,Z \quad \text{in particular:} \quad Y_{100} \simeq -14.4 + 5.4\,Z$$

Therefore

$$\begin{aligned}\mathbb{P}(\{S_{100} > S_0/10\}) &= \mathbb{P}(\{S_{100}/S_0 > 1/10\}) = \mathbb{P}(\{Y_{100} > -\log(10)) \\ &\simeq \mathbb{P}\left(\left\{Z > \frac{-2.3 + 14.4}{5.4}\right\}\right) = \mathbb{P}(\{Z > 2.24\})\end{aligned}$$

which is roughly 1% (so you can imagine what $\mathbb{P}(\{S_{100} > S_0\})$ looks like ...).

Therefore, the process $(S_n,\, n \geq 1)$, unexpectedly perhaps, "crashes" to zero with high probability as $n$ gets large, even though it seemed a priori a "fair game" with constant expectation. This is an important example among a large class of processes called "martingales"; we will come back to this!

*Note:* The random process $(S_n,\, n \geq 1)$ is not unrelated to the following *deterministic* process defined recursively as

$$x_0 \in \mathbb{N}^*, \quad x_{n+1} = \begin{cases} x_n/2 & \text{if } x_n \text{ is even} \\ 3x_n + 1 & \text{if } x_n \text{ is odd} \end{cases}$$

in which an even number gets multiplied by $1/2$ and an odd number gets approximately multiplied by $3/2$ (because it first gets multiplied by 3 and then necessarily divided by 2, as $3x_n + 1$ is even). So if you consider that even and odd numbers appear naturally with probability $1/2$, then the two processes have something in common. But in the deterministic case, one has no proof that the process ultimately reaches the value 1 as $n$ gets large: this is the famous Collatz conjecture, which remains unsolved until now.

**Exercise 2*.** a) let us compute $\mathbb{E}(S_n) = \sum_{j=1}^n \mathbb{E}(X_j^{(n)}) = n\,\frac{\lambda}{n} = \lambda$ and

$$\mathrm{Var}(S_n) = \sum_{j=1}^n \mathrm{Var}(X_j^{(n)}) = n\,\frac{\lambda}{n}\left(1 - \frac{\lambda}{n}\right) = \lambda - \frac{\lambda^2}{n}$$

b) So $\mu = \lim_{n\to\infty} \mathbb{E}(S_n) = \lambda$ and $\sigma^2 = \lim_{n\to\infty} \mathrm{Var}(S_n) = \lambda$.

c) Let us compute the characteristic function of $S_n$:

$$\begin{aligned}\phi_{S_n}(t) &= \mathbb{E}(\exp(itS_n)) = \mathbb{E}(\exp(it\,(X_1^{(n)} + \ldots + X_n^{(n)}))) = \mathbb{E}(\exp(itX_1^{(n)}))\cdots\mathbb{E}(\exp(itX_n^{(n)})) \\ &= \left(\mathbb{E}(\exp(itX_1^{(n)}))\right)^n = \left(e^{it}\,\frac{\lambda}{n} + 1 - \frac{\lambda}{n}\right)^n = \left(1 + \frac{\lambda\,(e^{it}-1)}{n}\right)^n \underset{n\to\infty}{\rightarrow} \exp\left(\lambda\,(e^{it}-1)\right)\end{aligned}$$

This limiting function is the characteristic function of $Z \sim \mathcal{P}(\lambda)$. Indeed, one can check that

$$\phi_Z(t) = \mathbb{E}(\exp(itZ)) = \sum_{k\geq 0} e^{itk}\,\frac{\lambda^k\,e^{-\lambda}}{k!} = e^{-\lambda}\sum_{k\geq 0}\frac{(\lambda\,e^{it})^k}{k!} = \exp(\lambda\,(e^{it}-1))$$

which allows us to conclude that $S_n \underset{n\to\infty}{\overset{d}{\rightarrow}} Z$.

d) The computation of the characteristic function is similar here:

$$\mathbb{E}\left(e^{itT_n}\right) = \left(\frac{1}{n}\,e^{it} + \left(1 - \frac{1}{n}\right)\right)^{\lceil \lambda n \rceil} = \left(1 + \frac{1}{n}\,(e^{it}-1)\right)^{\lceil \lambda n \rceil} \underset{n\to\infty}{\rightarrow} \exp(\lambda(e^{it}-1))$$

and leads actually exactly to the same result: $T_n$ converges in distribution towards a Poisson random variable $Z$ of parameter $\lambda$.

e) No, as each random variable $S_n$ is constructed from a different set of random variables $X_1^{(n)}, \ldots, X_n^{(n)}$, which depends on $n$. The same holds for the random variables $T_n$.