Homework 9

Exercise 1. Let $(m_k,\, k \geq 0)$ be the sequence of moments of a generic random variable X.

a) Let $\ell \geq k \geq 0$. Show that if $\mathbb{E}(|X|^\ell) < +\infty$ (i.e., if m_ℓ is well-defined and finite), then it also holds that $\mathbb{E}(|X|^k) < +\infty$.

Note: The reciprocal statement is that if $\ell \geq k \geq 0$ and $\mathbb{E}(|X|^k) = +\infty$, then $\mathbb{E}(|X|^\ell) = +\infty$.

b) Show that the growth of the odd moments is controlled by the growth of the even moments.

c) Show that if X is bounded, then Carleman's condition is satisfied:

$$\sum_{k \geq 1} m_{2k}^{-\frac{1}{2k}} = +\infty$$

Exercise 2. a) Let $X \sim \mathcal{N}(0,1)$. Compute all the moments of the random variable $Y = \exp(X)$.

b) Let W be the discrete random variable such that

$$\mathbb{P}(\{W = j\}) = C\, \exp(-j^2/2), \quad j \in \mathbb{Z}$$

where $C = 1/\sum_{j\in\mathbb{Z}} \exp(-j^2/2)$. Compute all the moments of the random variable $Z = \exp(W)$.

c) What can you conclude from parts a) and b)?

Exercise 3*. Let X be a $\mathcal{N}(0, \sigma^2)$ random variable, with $\sigma > 0$.

a) Using integration by parts, show that for any continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that there exists $C > 0$ and $n \geq 1$ with $|f(x)|, |f'(x)| \leq C\,(1 + x^2)^n$ for every $x \in \mathbb{R}$, we have

$$\mathbb{E}(X \cdot f(X)) = \sigma^2\, \mathbb{E}(f'(X))$$

Note: The above condition is needed to ensure that both expectations are finite.

b) Use part a) to compute $\mathbb{E}(X^k)$ for $k \in \mathbb{N}$.

Let now $m \geq 1$ and $Y = X^m$.

c) For which values of $m \geq 1$ and $\sigma > 0$ does it hold that the distribution of Y is entirely determined by its moments? (*Hint:* use Stirling's approximation: $k! \simeq k^k\, e^{-k}$).

Exercise 4. Let $(X_n,\, n \geq 1)$ be a sequence of i.i.d. non-negative random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and such that there exists $0 < a < b < +\infty$ with $a < X_n(\omega) \leq b$ for all $n \geq 1$ and $\omega \in \Omega$. Let also $(Y_n,\, n \geq 1)$ be the sequence defined as

$$Y_n = \left(\prod_{j=1}^{n} X_j \right)^{1/n}, \quad n \geq 1$$

a) Show that there exists a constant $\mu > 0$ such that $Y_n \underset{n\to\infty}{\to} \mu$ almost surely.

b) Compute the value of μ in the case where $\mathbb{P}(\{X_1 = a\}) = \mathbb{P}(\{X_1 = b\}) = \frac{1}{2}$ and $a, b > 0$.

c) In this case, look for the tightest possible upper bound on $\mathbb{P}(\{Y_n > t\})$ for $n \geq 1$ fixed and $t > \mu$.

Hint. You have two options here. One is to use Chebyshev's inequality with the function $\psi(x) = x^p$ and $p > 0$ (and then optimize over p) in order to upperbound

$$\mathbb{P}(\{Y_n > t\}) = \mathbb{P}\left(\left\{ \prod_{j=1}^{n} X_j > t^n \right\} \right)$$

for $t > \mu$. The other option is left to your imagination...