Solutions to Homework 9

Exercise 1. a) Using Jensen's inequality and the fact that the map $x \mapsto x^{\ell/k}$ is convex for $\ell \geq k$, we obtain

$$\mathbb{E}(|X|^\ell) = \mathbb{E}\left((|X|^k)^{\ell/k}\right) \geq (\mathbb{E}(|X|^k))^{\ell/k}$$

which leads to the result.

b) Two options here: i) By Cauchy-Schwarz' inequality, we obtain

$$|m_{2k+1}| \leq \mathbb{E}(|X|^{2k+1}) = \mathbb{E}(|X|^k\,|X|^{k+1}) \leq \sqrt{\mathbb{E}(X^{2k})\,\mathbb{E}(X^{2k+2})} = \sqrt{m_{2k}\,m_{2k+2}}$$

ii) Using the inequality $|a| \leq \frac{1+a^2}{2}$, we obtain

$$|m_{2k+1}| \leq \mathbb{E}(|X|^{2k+1}) \leq \frac{1}{2}\left(\mathbb{E}(|X|^{2k}) + \mathbb{E}(|X|^{2k+2})\right) = \frac{m_{2k} + m_{2k+2}}{2}$$

c) Saying that X is bounded amounts to saying that there exists $C > 0$ such that $|X(\omega)| \leq C$ for all $\omega \in \Omega$, so $|m_k| \leq \mathbb{E}(|X|^k) \leq \mathbb{E}(C^k) = C^k$. Therefore,

$$\sum_{k\geq 1} m_{2k}^{-\frac{1}{2k}} \geq \sum_{k\geq 1} (C^{2k})^{-\frac{1}{2k}} = \sum_{k\geq 1} \frac{1}{C} = +\infty$$

which proves the claim.

Exercise 2. a) We have

$$m_k = \mathbb{E}(Y^k) = \mathbb{E}(\exp(kX)) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{kx}\,\exp(-x^2/2)\,dx = \frac{1}{\sqrt{2\pi}}\,e^{k^2/2} \int_{\mathbb{R}} e^{-(x-k)^2/2}\,dx = e^{k^2/2}.$$

b) We have

$$m_k = \mathbb{E}(Z^k) = \mathbb{E}(\exp(kW)) = C \sum_{j\in\mathbb{Z}} e^{jk}\,e^{-j^2/2} = C\,e^{k^2/2} \sum_{j\in\mathbb{Z}} e^{-(j-k)^2/2} = e^{k^2/2}$$

c) So all the moments of Y and Z are identical, even though these two random variables clearly do not have the same distribution (one being continuous and the other being discrete). One can check indeed that the sequence of moments (m_k, $k \geq 0$) does not satisfy Carleman's condition, as

$$\sum_{k\geq 1} m_{2k}^{-\frac{1}{2k}} = \sum_{k\geq 1} e^{-k} < +\infty$$

Exercise 3*. a) Using integration by parts (with $u(x) = f(x)$ and $v'(x) = x\,\exp(-x^2/2\sigma^2)$), we obtain:

$$\mathbb{E}(X \cdot f(X)) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} x\,f(x)\,\exp(-x^2/2)\,dx = \frac{\sigma^2}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} f'(x)\,\exp(-x^2/2)\,dx = \sigma^2\,\mathbb{E}(f'(X))$$

(The assumptions made imply indeed that the boundary terms at $\pm\infty$ vanish, as an exponential decrease wins over a polynomial increase).

b) Using the above relation with $f(x) = x^{k-1}$, we obtain for $k \geq 2$:

$$\mathbb{E}(X^k) = \sigma^2 \, (k-1) \, \mathbb{E}(X^{k-2})$$

As $\mathbb{E}(X_1) = 0$, this implies that all odd moments equal zero (but this could also be directly deduced from the fact that for k odd, the function under the integral sign is odd). For the even moments, we obtain (using $\mathbb{E}(X^0) = 1$):

$$\mathbb{E}(X^k) = \sigma^{2k} \, (k-1) \, (k-3) \cdots 3 \cdot 1 = \sigma^k \, (k-1)!! = \sigma^k \, \frac{k!}{2^{k/2} \, (k/2)!}$$

c) Let us compute (using Stirling's approximation):

$$m_{2k} = \mathbb{E}(Y^{2k}) = \mathbb{E}(X^{2mk}) = \sigma^{2mk} \, \frac{2mk!}{2^{mk} \, (mk)!} \simeq \sigma^{2mk} \, \frac{(2mk)^{2mk} \, e^{-2mk}}{(2mk)^{mk} \, e^{-mk}}$$
$$= \sigma^{2mk} \, (2mk)^{mk} \, e^{-mk}$$

so $m_{2k}^{-1/(2k)} \simeq \sigma^{-m} \, (2mk)^{-m/2} \, e^{m/2}$ and therefore, Carleman's condition:

$$\sum_{k \geq 1} m_{2k}^{-1/(2k)} \simeq \sigma^{-m} \, (2m)^{-m/2} \, e^{m/2} \sum_{k \geq 1} k^{-m/2} = +\infty$$

is satisfied if and only if $m \leq 2$ ($\sigma > 0$ does not play a role here).

Exercise 4. a) Consider

$$\log(Y_n) = \frac{1}{n} \sum_{j=1}^{n} \log(X_j)$$

As $\log(X_j)$ are i.i.d. bounded random variables, the strong law of large numbers applies, so

$$\frac{1}{n} \sum_{j=1}^{n} \log(X_j) \underset{n \to \infty}{\to} \mathbb{E}(\log(X_1)) \quad \text{almost surely}$$

so $\mu = \exp(\mathbb{E}(\log(X_1)))$.

b) In this case $\mathbb{E}(\log(X_1)) = \frac{\log(a)+\log(b)}{2}$, so $\mu = \sqrt{ab}$.

c) Observing that $\log(X_j) \in [\log(a), \log(b)]$ and using (the generalized version of) Hoeffding's inequality, we obtain

$$\mathbb{P}(\{Y_n \geq t\}) = \mathbb{P}(\{\log(Y_n) - \log(\mu) \geq \log(t) - \log(\mu)\}) \leq \exp\left(-\frac{2n \, (\log(t) - \log(\mu))^2}{(\log(b) - \log(a))^2}\right)$$

so $\mathbb{P}(\{Y_n \geq t\}) \leq C^n$ for every $t > \mu = \sqrt{ab}$, and a possible value for C is $\exp(-\frac{2\,(\log(t)-\log(\mu))^2}{(\log(b)-\log(a))^2})$ (note that the same result may be obtained by a direct computation and the use of the inequality $\cosh(x) \leq \exp(x^2/2)$).

And a weaker result can be obtained also via the inequality

$$\mathbb{P}(\{Y_n \geq t\}) \leq \frac{\mathbb{E}(Y_n^n)}{t^n} = \left(\frac{\mathbb{E}(X_1)}{t}\right)^n = \left(\frac{a+b}{2t}\right)^n$$

which shows only that concentration holds for every $t > \frac{a+b}{2}$ (and not $t > \mu$).