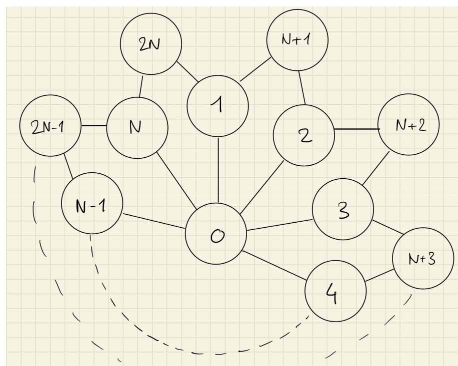
Final exam

Reminders: For $0 < p < 1$, $\sum_{k\geq 0} p^k = \frac{1}{1-p}$ and $\sum_{k\geq 1} k\, p^{k-1} = \frac{\partial}{\partial p}\left(\sum_{k\geq 1} p^k\right) = \frac{1}{(1-p)^2}$.

Exercise 1. (24 points) Let $N \geq 2$. Consider the Markov chain $(X_n,\, n \geq 0)$ with state space $S = \{0, 1, \ldots, 2N\}$ and transition graph:



where the transition probabilities are those given by a random walk evolving on the graph, i.e.

$$\begin{cases} p_{0,j} = \frac{1}{N} \quad \text{and} \quad p_{j,0} = \frac{1}{3} \quad \text{for } 1 \leq j \leq N \\[2ex] p_{1,N+1} = p_{1,2N} = \frac{1}{3} \quad \text{and} \quad p_{j,N+j-1} = p_{j,N+j} = \frac{1}{3} \quad \text{for } 2 \leq j \leq N \\[2ex] p_{2N,N} = p_{2N,1} = \frac{1}{2} \quad \text{and} \quad p_{N+j,j} = p_{N+j,j+1} = \frac{1}{2} \quad \text{for } 1 \leq j \leq N-1 \end{cases}$$

a) Explain why this chain admits a unique stationary distribution.

b) Compute the stationary distribution π of the chain.

Hint: Try detailed balance !

c) Is π also a limiting distribution for some values of $N \geq 2$? Justify.

d) Let $T_0 = \inf\{n \geq 1 \,:\, X_n = 0)$. Compute $f_{00}^{(n)} = \mathbb{P}(T_0 = n \,|\, X_0 = 0)$ for $n \geq 1$.

e) Compute $\mathbb{E}(T_0 \,|\, X_0 = 0)$. Does this value increase/decrease when N increases ?

BONUS f) For $N = 2$, what are the five eigenvalues of the transition matrix P ?

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Exercise 2. (20 points) Let $p \geq 0$ and consider a Markov chain $(X_n,\, n \geq 0)$ on the state space $S = \{0, 1, 2, 3\}$ with transition probabilities

$$p_{i,i+1} = 2p \quad \text{and} \quad p_{i+1,i} = p \quad \text{for } i \in \{0,1,2\}$$

a) Complete the transition matrix P (exclusively with self-loops). What are the possible values taken by the parameter $p \geq 0$?

b) For what values of $p \geq 0$ is the chain X ergodic ? Justify your answer.

From now on, we will assume that the value of p is such that the chain X is ergodic.

c) Compute the limiting and stationary distribution π of the chain X.

d) Compute the spectral gap γ of the chain. For what value of p is γ maximal ?

Hint: The eigenvalues of the matrix $A = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & -1 & 1 \end{pmatrix}$ are equal to $0, 1, 3, 5$, respectively.

e) For what value of $i \in S$ is $\|P_i^n - \pi\|_{\mathrm{TV}}$ the largest when n gets large ?

Exercise 3. (16 points) Let d be a positive integer and $S = \{0,1\}^d$. Consider the following distribution π on S:

$$\pi_x = \frac{2^{-|x|}}{Z}, \quad x \in S$$

where $|x| = \sum_{i=1}^d x_i$ is the Hamming weight of state x (and Z is the normalization constant).

a) Starting from the base chain which is the symmetric random walk on the hypercube (with no self loop), construct the Metropolis chain whose limiting and stationary distribution is equal to π.

In particular, give the explicit formula for the transition matrix P of the Metropolis chain; what is the value of p_{xx} for $x \in S$?

b) In this case, the value of Z can be computed exactly for any $d \geq 1$. Do it !