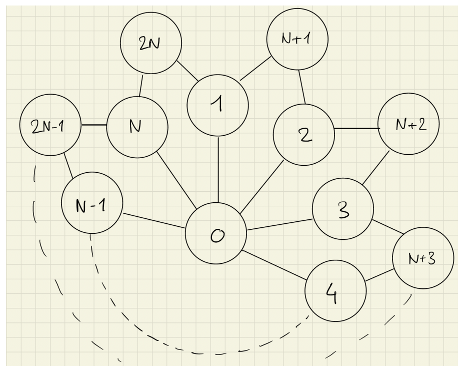
Final exam: solutions

Reminders: For $0 < p < 1$, $\sum_{k\geq 0} p^k = \frac{1}{1-p}$ and $\sum_{k\geq 1} k\, p^{k-1} = \frac{\partial}{\partial p}\left(\sum_{k\geq 1} p^k\right) = \frac{1}{(1-p)^2}$.

Exercise 1. (24 points) Let $N \geq 2$. Consider the Markov chain $(X_n,\, n \geq 0)$ with state space $S = \{0, 1, \ldots, 2N\}$ and transition graph:



where the transition probabilities are those given by a random walk evolving on the graph, i.e.

$$\begin{cases} p_{0,j} = \frac{1}{N} \quad \text{and} \quad p_{j,0} = \frac{1}{3} \quad \text{for } 1 \leq j \leq N \\[2ex] p_{1,N+1} = p_{1,2N} = \frac{1}{3} \quad \text{and} \quad p_{j,N+j-1} = p_{j,N+j} = \frac{1}{3} \quad \text{for } 2 \leq j \leq N \\[2ex] p_{2N,N} = p_{2N,1} = \frac{1}{2} \quad \text{and} \quad p_{N+j,j} = p_{N+j,j+1} = \frac{1}{2} \quad \text{for } 1 \leq j \leq N-1 \end{cases}$$

a) (4 pts) Explain why this chain admits a unique stationary distribution.

Answer: The chain is irreducible and finite, therefore positive-recurrent, so by the theorem seen in class, it admits a unique stationary distribution.

b) (8 pts) Compute the stationary distribution π of the chain.

Hint: Try detailed balance !

Answer: With detailed balance, we obtain that $\pi_0 \cdot \frac{1}{N} = \pi_i \cdot \frac{1}{3}$ and $\pi_i \cdot \frac{1}{3} = \pi_j \cdot \frac{1}{2}$ for $1 \leq i \leq N$ and $N+1 \leq j \leq 2N$ such that i and j are connected. This gives

$$\pi = (N, 3, \ldots, 3, 2, \ldots 2)/Z \quad \text{where} \quad Z = N + 3N + 2N = 6N$$

c) (3 pts) Is π also a limiting distribution for some values of $N \geq 2$? Justify.

Answer: No: the chain is 2-periodic for every value of $N \geq 2$.

d) (5 pts) Let $T_0 = \inf\{n \geq 1 \,:\, X_n = 0\}$. Compute $f_{00}^{(n)} = \mathbb{P}(T_0 = n \,|\, X_0 = 0)$ for $n \geq 1$.

Answer: $f_{00}^{(2n)} = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1}$ and $f_{00}^{(2n-1)} = 0$ for $n \geq 1$.

e) (4 pts) Compute $\mathbb{E}(T_0 \,|\, X_0 = 0)$. Does this value increase/decrease when N increases ?

Answer: Two possibilities: the simplest is to use the fact that $\mathbb{E}(T_0 \,|\, X_0 = 0) = \frac{1}{\pi_0} = 6$. The other one is to compute

$$\mathbb{E}(T_0 \,|\, X_0 = 0) = \sum_{n\geq 1} n \cdot f_{00}^{(n)} = \sum_{n\geq 1} 2n \cdot f_{00}^{(2n)} = \sum_{n\geq 1} n \cdot \left(\frac{2}{3}\right)^n = 6$$

using the hint. So the value of $\mathbb{E}(T_0 \,|\, X_0 = 0)$ does not depend on N, perhaps surprisingly.

BONUS f) (3 pts) For $N = 2$, what are the five eigenvalues of the transition matrix P ?

Answer: In this case, the matrix P is given by

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \end{pmatrix}$$

It has rank 2 and trace 0, and we know that one of its eigenvalues is 1, so its eigenvalues are given by $(1, 0, 0, 0, -1)$. (Note that we know also that one of its eigenvalues is -1 because the chain is 2-periodic).

Exercise 2. (20 points) Let $p \geq 0$ and consider a Markov chain $(X_n,\, n \geq 0)$ on the state space $S = \{0, 1, 2, 3\}$ with transition probabilities

$$p_{i,i+1} = 2p \quad \text{and} \quad p_{i+1,i} = p \quad \text{for } i \in \{0, 1, 2\}$$

a) (4 pts) Complete the transition matrix P (exclusively with self-loops). What are the possible values taken by the parameter $p \geq 0$?

Answer: The resulting graph corresponds to the transition matrix

$$P = \begin{pmatrix} 1-2p & 2p & 0 & 0 \\ p & 1-3p & 2p & 0 \\ 0 & p & 1-3p & 2p \\ 0 & 0 & p & 1-p \end{pmatrix}$$

Accordingly, p must lie in the interval $[0, 1/3]$.

b) (4 pts) For what values of $p \geq 0$ is the chain X ergodic ? Justify your answer.

Answer: Only for $0 < p \leq 1/3$: in this case, the chain is finite and irreducible, so positive-recurrent, and also aperiodic (due to the presence of non-zero self-loops), therefore ergodic.

From now on, we will assume that the value of p is such that the chain X is ergodic.

c) (5 pts) Compute the limiting and stationary distribution π of the chain X.

Answer: By detailed balance again, we have $\pi_i \cdot 2p = \pi_{i+1} \cdot p$, so $\pi_{i+1} = 2 \cdot \pi_i$ for $i \in \{0, 1, 2\}$, i.e.

$$\pi = (1, 2, 4, 8)/Z \quad \text{where} \quad Z = 15$$

d) (5 pts) Compute the spectral gap γ of the chain. For what value of p is γ maximal ?

Hint: The eigenvalues of the matrix $A = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & -1 & 1 \end{pmatrix}$ are equal to $0, 1, 3, 5$, respectively.

Answer: Observe that $P = I - pA$, so the eigenvalues of P are given by $(1, 1-p, 1-3p, 1-5p)$, and the spectral gap $\gamma = \min(p, 2-5p)$, which is maximal when $p = 2-5p$, i.e. $p = 1/3$ (in which case $\gamma = 1/3$ also).

e) (2 pts) For what value of $i \in S$ is $\|P_i^n - \pi\|_{\mathrm{TV}}$ the largest when n gets large ?

Answer: State 0, because it is the state with the smallest probability with respect to the stationary distribution π.

Exercise 3. (16 points) Let d be a positive integer and $S = \{0, 1\}^d$. Consider the following distribution π on S:

$$\pi_x = \frac{2^{-|x|}}{Z}, \quad x \in S$$

where $|x| = \sum_{i=1}^d x_i$ is the Hamming weight of state x (and Z is the normalization constant).

a) (12 pts) Starting from the base chain which is the symmetric random walk on the hypercube (with no self loop), construct the Metropolis chain whose limiting and stationary distribution is equal to π.

In particular, give the explicit formula for the transition matrix P of the Metropolis chain; what is the value of p_{xx} for $x \in S$?

Answer: The base chain has transition probabilities $\psi_{xy} = \frac{1}{d}$ for $y \sim x$ (and 0 otherwise), and the acceptance probabilities are given by

$$a_{xy} = \min(1, \pi_y/\pi_x) = \begin{cases} 1 & \text{if } y \sim x,\ |y| = |x| - 1 \\ 1/2 & \text{if } y \sim x,\ |y| = |x| + 1 \end{cases}$$

So the transition probabilities of the Metropolis chain are given by

$$p_{xy} = \begin{cases} 1/d & \text{if } y \sim x,\ |y| = |x| - 1 \\ 1/2d & \text{if } y \sim x,\ |y| = |x| + 1 \end{cases}$$

$$p_{xx} = \sum_{y \sim x \,:\, |y| = |x|+1} \frac{1}{2d} = \frac{d - |x|}{2d}$$

and 0 otherwise. One checks indeed that $\sum_{y \in S} p_{xy} = 1$ for every $x \in S$.

b) (4 pts) In this case, the value of Z can be computed exactly for any $d \geq 1$. Do it !

Answer: $Z = \sum_{x \in S} 2^{-|x|} = \sum_{x_1, \ldots, x_d \in \{0,1\}} 2^{-(x_1 + \ldots + x_d)} = \left(\sum_{x_1 \in \{0,1\}} 2^{-x_1} \right)^d = (3/2)^d$