Homework 4

Exercise 1*. Let $\lambda > 0$ and X be a random variable whose characteristic function ϕ_X is given by $\phi_X(t) = \exp(-\lambda|t|), \quad t \in \mathbb{R}$.

a) What can you deduce on the distribution of X from each of the following facts?

i) ϕ_X is not differentiable in $t = 0$.

ii) $\int_{\mathbb{R}} |\phi_X(t)|\, dt < +\infty$.

b) Use the inversion formula seen in class to compute the distribution of X.

c) Let $Y = \frac{1}{X}$. Using the change of variable formula (not worrying about the fact that X might take the value 0, as this is a negligible event), compute the distribution of Y.

d) Let now $X_1, \ldots, X_n$ be n independent copies of the random variable X. What are the distributions of

$$Z_n = \frac{X_1 + \ldots + X_n}{n} \quad \text{and} \quad W_n = \frac{n}{\frac{1}{X_1} + \ldots + \frac{1}{X_n}} \quad ?$$

e) What oddities do you observe in the results of part d)? (there are at least two)

Exercise 2. a) Let X_1, X_2 be two independent Gaussian random variables such that $\mathrm{Var}(X_1) = \mathrm{Var}(X_2)$. Show, using characteristic functions or a result from the course, that X_1+X_2 and X_1-X_2 are also independent Gaussian random variables.

b) Let X_1, X_2 be two independent square-integrable random variables such that $X_1 + X_2, X_1 - X_2$ are also independent random variables. Show that X_1, X_2 are jointly Gaussian random variables such that $\mathrm{Var}(X_1) = \mathrm{Var}(X_2)$.

Note. Part b), also known as Darmois-Skitovic's theorem, is considerably more challenging than part a)! Here are the steps to follow in order to prove the result (but please skip the first two).

Step 1.* (needs the dominated convergence theorem, which is outside of the scope of this course) If X is a square-integrable random variable, then ϕ_X is twice continuously differentiable.

Step 2.* (quite technical) Under the assumptions made, ϕ_{X_1} and ϕ_{X_2} have no zeros (so $\log \phi_{X_1}$ and $\log \phi_{X_2}$ are also twice continuously differentiable, according to the previous step).

Step 3. Let $f_1 = \log \phi_{X_1}$ and $f_2 = \log \phi_{X_2}$. Show that there exist functions g_1, g_2 satisfying

$$f_1(t_1 + t_2) + f_2(t_1 - t_2) = g_1(t_1) + g_2(t_2) \quad \forall t_1, t_2 \in \mathbb{R}$$

Step 4. If f_1, f_2 are twice continuously differentiable and there exist functions g_1, g_2 satisfying

$$f_1(t_1 + t_2) + f_2(t_1 - t_2) = g_1(t_1) + g_2(t_2) \quad \forall t_1, t_2 \in \mathbb{R}$$

then f_1, f_2 are polynomials of degree less than or equal to 2. *Hint:* differentiate!

Step 5. If X is square-integrable and $\log \phi_X$ is a polynomial of degree less than or equal to 2, then X is a Gaussian random variable.

Hint. If X is square-integrable, then you can take for granted that $\phi_X(0) = 1$, $\phi'_X(0) = i\mathbb{E}(X)$ and $\phi''_X(0) = -\mathbb{E}(X^2)$.

Step 6. From the course, deduce that X_1, X_2 are jointly Gaussian and that $\mathrm{Var}(X_1) = \mathrm{Var}(X_2)$.