APA lecture 8a

Convergence in distribution:

$$X_n \xrightarrow{d} X \quad \text{if} \quad F_{X_n}(t) \longrightarrow F_X(t)$$

$$\forall t \in \mathbb{R} \text{ s.t. } t = \text{continuity point of } F_X$$

Proposition:

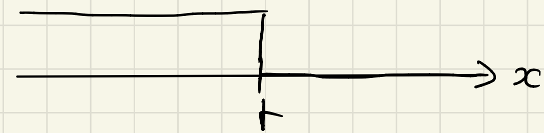
$$X_n \xrightarrow{d} X \quad \text{iff} \quad \mathbb{E}(g(X_n)) \longrightarrow \mathbb{E}(g(X))$$

$$\forall g \in C_b(\mathbb{R})$$

Two remarks

- For fixed $t \in \mathbb{R}$, define the function $h_t(x) = 1_{\{x \leq t\}}$

  Then $F_X(t) = \mathbb{E}(h_t(X))$

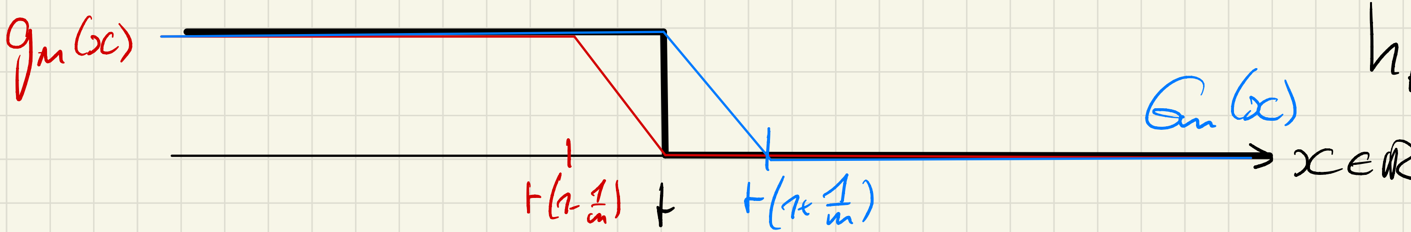


- If X, Y are r.v.'s st $\mathbb{E}(g(X)) = \mathbb{E}(g(Y))$ $\forall g \in C_b(\mathbb{R})$, then $X \sim Y$

  so knowing $\mathbb{E}(g(X))$ $\forall g$ characterizes fully the distribution of X.

Proof: ("if" part: if $\mathbb{E}(g(X_n)) \to \mathbb{E}(g(X))$ $\forall g \in C_b(\mathbb{R})$
then $F_{X_n}(t) \to F_X(t)$ $\forall t = \ldots$)

(fix m for now)



$h_t(x) = 1_{x \leq t}$

Define $g_m(x) = \begin{cases} 1 & \text{if } x \leq t(1-\frac{1}{m}) \\ m(1-\frac{x}{t}) & \text{if } t(1-\frac{1}{m}) \leq x \leq t \\ 0 & \text{if } x \geq t \end{cases}$

$$G_m(x) = \begin{cases} 1 & \text{if } x \leq t \\ \ldots & \text{if } t \leq x \leq t(1+\frac{1}{m}) \\ 0 & \text{if } x \geq t(1+\frac{1}{m}) \end{cases}$$

By the assumption made $(g_m, G_m \in C_b(\mathbb{R}))$:

$$\mathbb{E}(g_m(X_n)) \xrightarrow[n\to\infty]{} \mathbb{E}(g_m(X)) \quad \forall m \geq 1$$

$$\mathbb{E}(G_m(X_n)) \xrightarrow[n\to\infty]{} \mathbb{E}(G_m(X)) \quad \forall m \geq 1$$

Also $g_m(x) \leq h_t(x) \leq G_m(x) \quad \forall m \geq 1, \forall x \in \mathbb{R}$

$$\mathbb{E}(g_m(X)) = \lim_{n\to\infty} \mathbb{E}(g_m(X_n)) \leq \lim_{n\to\infty} \mathbb{E}(h_t(X_n)) \text{ (if it exists)}$$

$$\leq \lim_{n\to\infty} \mathbb{E}(G_m(X_n)) = \mathbb{E}(G_m(X))$$

Monotone convergence thm (not seen in this class)

- $g_m(x) \le g_{m+1}(x)$

  so $\lim_{m\to\infty} \mathbb{E}(g_m(X)) = \mathbb{E}(\lim_{m\to\infty} g_m(X))$

  $= \mathbb{E}(\mathbb{1}_{\{X<t\}}) = \mathbb{P}(\{X<t\})$

- $G_m(x) \ge G_{m+1}(x)$

  so $\lim_{m\to\infty} \mathbb{E}(G_m(X)) = \mathbb{E}(\lim_{m\to\infty} G_m(X))$

  $= \mathbb{E}(\mathbb{1}_{\{X\le t\}}) = \mathbb{P}(\{X\le t\})$

Conclusion:

(if it exists)

$$\mathbb{P}(\{X<t\}) \leq \lim_{n\to\infty} \mathbb{E}(h_t(X_n)) \leq \mathbb{P}(\{X\leq t\})$$

if t = continuity point of F_X, then LHS = RHS

and we obtain convergence:

$$\lim_{n\to\infty} F_{X_n}(t) = F_X(t)$$

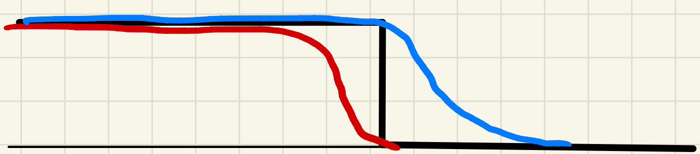
#

<u>Proposition</u>

$X_n \xrightarrow{d} X$ iff $\mathbb{E}(g(X_n)) \to \mathbb{E}(g(X))$

$\forall g \in C_b^3(\mathbb{R})$ i.e. g is 3 times continuously differentiable and g, g', g'', g''' are bounded

<u>"Pf"</u>:



- **Proposition**

$$X_n \xrightarrow{d} X \quad \text{iff} \quad \phi_{X_n}(t) \longrightarrow \phi_X(t) \quad \underline{\underline{\forall t \in \mathbb{R}}}$$

$$\left(\text{here} \quad g_t(x) = e^{itx}, \quad t \in \mathbb{R} \right)$$

- Characterization of convergence with moments: $g_k(x) = x^k$, $k \geq 1$ ⚠ some details missing

The central limit theorem

Let $(X_n, n \geq 1)$ be iid r.v. st $\mathbb{E}(X_1^2) < +\infty$

$$S_n = X_1 + \dots + X_n \quad n \geq 1$$

SLLN : $\frac{S_n}{n} \xrightarrow[n\to\infty]{} \mu = \mathbb{E}(X_1)$ a.s.

Some simple computations:

$$\mathbb{E}(S_n) = n \cdot \mathbb{E}(X_1) = n \cdot \mu$$

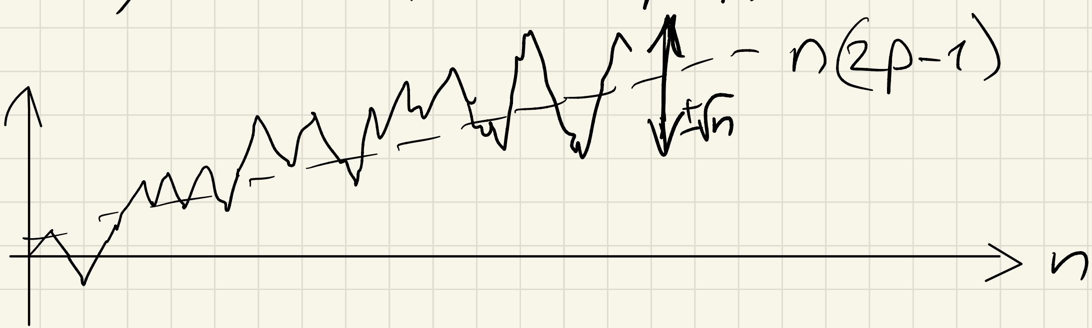
$$Var(S_n) = n \cdot Var(X_1) = n \sigma^2$$

Define $\tilde{S}_n = \dfrac{S_n - n\mu}{\sqrt{n}\,\sigma}$ (equiv: $S_n = n\mu + \sqrt{n}\,\sigma\,\tilde{S}_n$)

Observe that $\mathbb{E}(\tilde{S}_n) = 0$ and $\mathrm{Var}(\tilde{S}_n) = 1$ $\forall n$

So "loosely speaking": $S_n = n\mu + \sqrt{n}\,\sigma\,\underbrace{\tilde{S}_n}_{\text{mean } 0,\ \text{var } 1}$

For ex, take $X_1 = \pm 1$ w.p. $p, 1-p$



CLT: $\tilde{S}_n \xrightarrow{d} Z \sim N(0,1)$

$\forall$ distribution of X_1 satisfying $\mathbb{E}(X_1^2) < +\infty$

= universal result

Rmks: • i.e. $\mathbb{P}(\{\tilde{S}_n \leq t\}) \xrightarrow[n\to\infty]{} \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \quad \forall t \in \mathbb{R}$

- SLLN: strong convergence vs very approximate result
- CLT: weak convergence vs more precise result
- for the proof, we will assume $\mathbb{E}(|X_1|^3) < +\infty$.
- but note here that we cannot skip $\mathbb{E}(X_1^2) < +\infty$.

"Second" proof (relying on characteristic functions)

$$\tilde{S}_n \xrightarrow{d} \underset{\sim N(0,1)}{Z} \quad \text{iff} \quad \phi_{\tilde{S}_n}(t) \underset{n\to\infty}{\longrightarrow} \phi_Z(t) = e^{-t^2/2}$$

$\forall t \in \mathbb{R}$

Let us compute:

$$\phi_{\tilde{S}_n}(t) = \mathbb{E}\left(e^{it\tilde{S}_n}\right) = \mathbb{E}\left(e^{it(S_n - n\mu)/\sqrt{n}\sigma}\right)$$

$$= \mathbb{E}\left(\exp\left(\frac{it}{\sqrt{n}\sigma}\left(\sum_{j=1}^{n} X_j - n\mu\right)\right)\right)$$

$$= \mathbb{E}\left(\exp\left(\frac{it}{\sqrt{n}\sigma}\left(\sum_{j=1}^{n}(X_j - \mu)\right)\right)\right)$$

$$= \mathbb{E}\left(\prod_{j=1}^{n} \exp\left(\frac{it}{\sqrt{n}\cdot\sigma}(X_j - \mu)\right)\right)$$

Taylor: $\exp(x) = 1 + x + \frac{x^2}{2} + O(|x|^3)$

small x

By independence:

$$\phi_{\tilde{S}_n}(t) = \prod_{j=1}^{n} \mathbb{E}\left(\exp\left(\frac{it}{\sqrt{n}\sigma}(X_j-\mu)\right)\right)$$

$$\overset{\text{id.}}{=} \mathbb{E}\left(\exp\left(\frac{it}{\sqrt{n}\sigma}(X_1-\mu)\right)\right)^n$$

random var.

$$= \left(\mathbb{E}\left(1 + \frac{it}{\sqrt{n}\sigma}(X_1-\mu) - \frac{t^2}{2n\sigma^2}(X_1-\mu)^2 + O\left(\frac{|X_1|^3}{n^{3/2}}\right)\right)\right)^n$$

$$= \left(1 + \frac{it}{\sqrt{n}\sigma}\underbrace{\mathbb{E}(X_1-\mu)}_{=0} - \frac{t^2}{2n\sigma^2}\cdot\underbrace{\mathbb{E}\left((X_1-\mu)^2\right)}_{=\sigma^2} + O\left(\frac{1}{n^{3/2}}\right)\right)^n$$

$$= \left(1 - \frac{t^2}{2n} + O\left(\frac{1}{n^{3/2}}\right)\right)^n \xrightarrow[n\to\infty]{} \exp\left(-\frac{t^2}{2}\right)$$