Midterm exam

Please pay attention to the presentation of your answers! **(2 points)**

Exercise 1. Quiz. (18 points)

Answer each yes/no question below (1 pt) and provide a short justification (proof or counter-example) for your answer (2 pts).

a) Let X, Y be two random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G} = \sigma(X) \cap \sigma(Y)$ [fact: it can be shown that $\mathcal{G}$ is a σ-field]. Is it true that $\{X \leq Y\} \in \mathcal{G}$?

b) Let X, Y be two independent random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Is it always true that $\sigma(X + Y) = \sigma(X, Y)$?

c) Let X be a continuous random variable whose pdf p_X is a continuous function on $\mathbb{R}$. Let now $Y = X^2$. Is it always true that the pdf p_Y is also a continuous function on $\mathbb{R}$?

d) Let F be a generic cdf. Is it always true that the function $G : \mathbb{R} \to [0, 1]$ defined as

$$G(t) = F(t^3 + 3t^2 + 3t + 1), \quad t \in \mathbb{R}$$

is also a cdf ?

e) Let X, Y, Z be three square-integrable random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, each with variance 2. Is it possible that $\text{Cov}(X, Y) = \text{Cov}(X, Z) = \text{Cov}(Y, Z) = -1$?

f) Let $(X_n, n \geq 1)$ and $(Y_n, n \geq 1)$ be two sequences of random variables that both converge in probability to the same random variable X. Is it always true that $X_n - Y_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$?

Exercise 2. (15 points)

Let X, Y be two i.i.d. $\mathcal{N}(0, 1)$ random variables, and Z be independent of X, Y and such that $\mathbb{P}\{Z = +1\}) = \mathbb{P}\{Z = -1\}) = 1/2$.

a) $(X + Z\,Y, Y)$ is a continuous random vector: compute its joint pdf.

b) Is it true $X + Z\,Y$ is a Gaussian random variable ? Justify.

c) Is it true $(X + Z\,Y, Y)$ is a Gaussian random vector ? Justify.

d) Compute $\text{Cov}(X + ZY, Y)$.

e) Is it true that $X + Z\,Y$ and Y are independent random variables ? Justify.

Exercise 3. (15 + 3 points)

Hint for this exercise (not necessarily needed): For $x \in \mathbb{R}$, $\exp(x) = \lim_{n\to\infty}(1 + x/n)^n$.

Let X be a random variable whose characteristic function is given by $\phi_X(t) = \max(1 - |t|, 0)$ for $t \in \mathbb{R}$.

Fact: ϕ_X is a characteristic function: we do not ask you to prove it.

a) Is X a continuous random variable ?

b) What is the value of $\mathbb{E}(|X|)$ and $\mathbb{E}(X^2)$?

Hint: You do not need to compute the distribution of X in order to answer the two previous questions, but please justify your answers !

Let now $(X_n,\, n \geq 1)$ be a sequence of i.i.d. random variables such that $X_n \sim X$ for every $n \geq 1$.

c) For $n \geq 1$, define $Y_n = \dfrac{X_1 + X_2 + \ldots + X_n}{n}$. Compute the characteristic function of Y_n.

d) Let $n \geq 1$ be fixed. For what values of $t \in \mathbb{R}$ does it hold that $\phi_{Y_n}(t) = 0$?

e) Does there exist $\mu \in \mathbb{R}$ such that $Y_n \underset{n\to\infty}{\to} \mu$ almost surely ? Justify.

BONUS f*) Compute the distribution of X.

Exercise 4. (10 points)

Hint for this exercise: For $X \sim \mathcal{N}(0, 1)$ and $t \geq 0$, it holds that $F_X(t) \geq 1 - \exp(-t^2/2)$.

Let $(\sigma_n,\, n \geq 1)$ be a sequence of positive numbers and $(Z_n,\, n \geq 1)$ be a sequence of independent random variables such that $Z_n \sim \mathcal{N}(0, \sigma_n^2)$. Let also $\mu \in \mathbb{R}$ and $X_n = \mu + Z_n$ for $n \geq 1$.

a) Show that if $\sigma_n \underset{n\to\infty}{\to} 0$, then $X_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} \mu$.

b) Assume now that $\sigma_n = \frac{1}{\log(n+1)}$ for $n \geq 1$. Is it true in this case that $X_n \underset{n\to\infty}{\to} \mu$ almost surely ? If yes, prove it; if no, explain why.

c) Does any of the conclusions of parts a) and b) rely on the fact that the random variables Z_n are independent ? Explain.