APA lecture 13a

Doob's martingale

Def: Consider X an integrable r.v.

& $(\mathcal{F}_n, n \in \mathbb{N})$ a filtration

$$M_n = \mathbb{E}(X \mid \mathcal{F}_n), \quad n \in \mathbb{N}$$

Prop: M is a martingale w.r.t. $(\mathcal{F}_n, n \in \mathbb{N})$

Proof: (i) $\mathbb{E}(|M_n|) = \mathbb{E}(|\mathbb{E}(X \mid \mathcal{F}_n)|) \underset{\text{Jensen}}{\leq} \mathbb{E}(\mathbb{E}(|X| \mid \mathcal{F}_n))$

$$= \mathbb{E}(|X|) < +\infty \quad \forall n \in \mathbb{N}$$

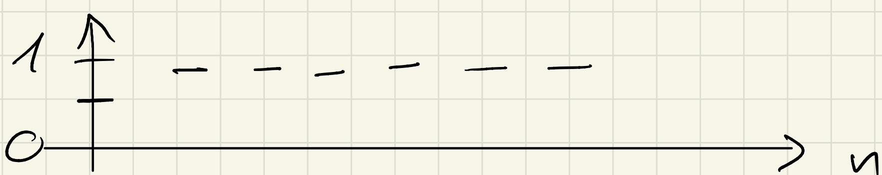
(ii) $M_n = \mathbb{E}(X \mid \mathcal{F}_n)$ is $\mathcal{F}_n$-measurable by def $\forall n \in \mathbb{N}$

(iii) $\mathbb{E}(M_{n+1} \mid \mathcal{F}_n) = \mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n)$ $\qquad \mathcal{F}_n \subset \mathcal{F}_{n+1}$

$$\underset{\text{towering property}}{=} \mathbb{E}(X \mid \mathcal{F}_n) = M_n \qquad \#$$

<u>Ex:</u> Take $X \sim U[0,1]$

$\mathcal{F}_n$ = information about the first n decimals of X



What about $\lim_{n\to\infty} M_n$? $\longrightarrow X$ a.s.

Martingale convergence theorem (v1)

Let $(M_n, n \in \mathbb{N})$ be a square-integrable martingale (i.e. $\mathbb{E}(M_n^2) < +\infty \ \forall n \in \mathbb{N}$) with respect to a filtration $(\mathcal{F}_n, n \in \mathbb{N})$, such that

$$\boxed{\sup_{n \in \mathbb{N}} \mathbb{E}(M_n^2) < +\infty}$$

(NB: in general, $\mathbb{E}(M_n^2)$ is an increasing fn because M^2 = submartingale)

Then there exists a random variable M_∞ s.t.

(i) $M_n \xrightarrow[n\to\infty]{} M_\infty$ a.s.

(ii) $M_n \xrightarrow[n\to\infty]{L^2} M_\infty$ (i.e. $\mathbb{E}\left((M_n - M_\infty)^2\right) \xrightarrow[n\to\infty]{} 0$)

(iii) $\mathbb{E}(M_\infty \mid \mathcal{F}_n) = M_n \quad \forall n \in \mathbb{N}$

"The martingale is closed at $+\infty$"

Ex: If X is square-integrable, then $M_n = \mathbb{E}(X \mid \mathcal{F}_n)$ satisfies the assumption, as

$$\mathbb{E}(M_n^2) = \mathbb{E}\left(\left(\mathbb{E}(X \mid \mathcal{F}_n)\right)^2\right) \overset{\text{Jensen}}{\leq} \mathbb{E}\left(\mathbb{E}(X^2 \mid \mathcal{F}_n)\right)$$

$$= \mathbb{E}(X^2) < +\infty$$

So $\sup_{n \in \mathbb{N}} \mathbb{E}(M_n^2) \leq \mathbb{E}(X^2)$

So by the thm, $\exists\, M_\infty$ s.t. (i) $M_n \longrightarrow M_\infty$

(ii) ... (iii) ...

but M_∞ is not nec. $= X$

Remarks

- If it holds that $\sup_{n\in\mathbb{N}} \sup_{\omega\in\Omega} |M_n(\omega)| \leq K < +\infty$ then in this $\sup_{n\in\mathbb{N}} \mathbb{E}(M_n^2) \leq K^2 < +\infty$.

- ⚠ The assumption $\sup_{n\in\mathbb{N}} \mathbb{E}(M_n^2) < +\infty$ might be replaced by the weaker assumption $\sup_n \mathbb{E}(|M_n|^{1+\varepsilon}) < +\infty$ for $\varepsilon > 0$ but <u>not</u> by $\sup_n \mathbb{E}(|M_n|) < +\infty$!

Example

Define $M_0 = x \in]0,1[$, $M_{n+1} = \begin{cases} M_n^2 & \text{wp } 1/2 \\ 2M_n - M_n^2 & \text{wp } 1/2 \end{cases}$

M is martingale: if $0 \leq M_n \leq 1$, then M_{n+1} either $= M_n^2 \in [0,1]$

or $= 2M_n - M_n^2 = 1 - 1 + 2M_n - M_n^2 = 1 - (1-M_n)^2 \in [0,1]$

so $\sup\limits_{n \in \mathbb{N}} \sup\limits_{\omega \in \Omega} |M_n(\omega)| \leq 1 < +\infty$

$\hookrightarrow \mathbb{E}(M_{n+1} \mid \underbrace{\mathcal{F}_n}_{\text{natural filtration}}) = \frac{1}{2} M_n^2 + \frac{1}{2}(2M_n - M_n^2) = M_n$ ✓

$$M_{n+1} = \begin{cases} M_n^2 & \text{wp } \frac{1}{2} \\ 1-(1-M_n)^2 & \text{wp } \frac{1}{2} \end{cases}$$

By the thm, $\exists M_\infty$ st

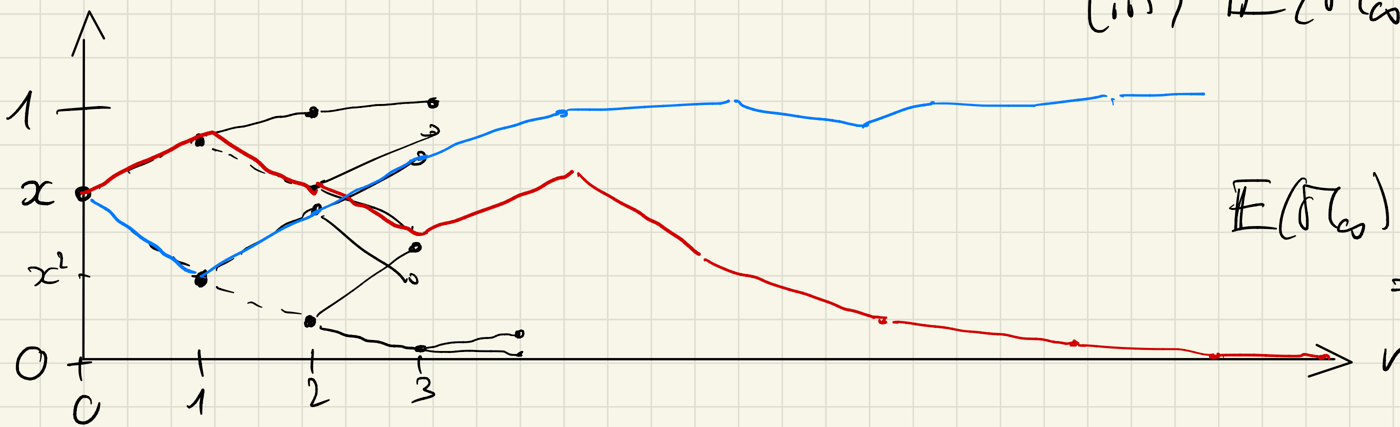
(i) $M_n \to M_\infty$ a.s.

(ii) $M_n \xrightarrow{L^2} M_\infty$

(iii) $\mathbb{E}(M_\infty \mid \mathcal{F}_n) = M_n \quad \forall n$

$\Downarrow$

$\mathbb{E}(M_\infty) = \mathbb{E}(M_n)$
$= \mathbb{E}(M_0) = x$



Actually, M_∞ takes only two possible values: 0 or 1

& $\mathbb{P}(\{M_\infty = 1\}) = x = 1 - \mathbb{P}(\{M_\infty = 0\})$

Consequences of the theorem

Optional stopping theorem, v2

Let M be a sq-int. martingale wrt a filtration $(\mathcal{F}_n, n \in \mathbb{N})$ satisfying $\sup_{n \in \mathbb{N}} \mathbb{E}(M_n^2) < +\infty$

Let T_1, T_2 be two stopping times wrt $(\mathcal{F}_n, n \in \mathbb{N})$ st.

$$0 \leq T_1(\omega) \leq T_2(\omega) \leq +\infty \quad \forall \omega \in \Omega$$

Then $\mathbb{E}(M_{T_2} \mid \mathcal{F}_{T_1}) = M_{T_1}$ a.s. & $\mathbb{E}(M_{T_2}) = \mathbb{E}(M_{T_1})$

Proof:

Redo the proof of the optional stopping theorem, v1 with $N = +\infty$ (& use MCT).

Still....

We cannot yet apply the thm for $M = S$ classical RW, and T_1, T_2 unbounded.

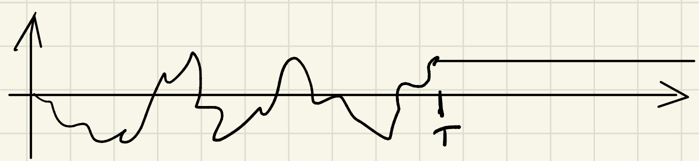
Def: <u>stopped martingale</u>

Let M be a martingale w.r.t $(\mathcal{F}_n, n \in \mathbb{N})$

T be a stopping time " "

Define the new process $M^T = (M^T_n, n \in \mathbb{N})$

as $M^T_n = M_{T \wedge n} \equiv M_{\min(T,n)}$



Prop: M^T is a martingale w.r.t $(\mathcal{F}_n, n \in \mathbb{N})$

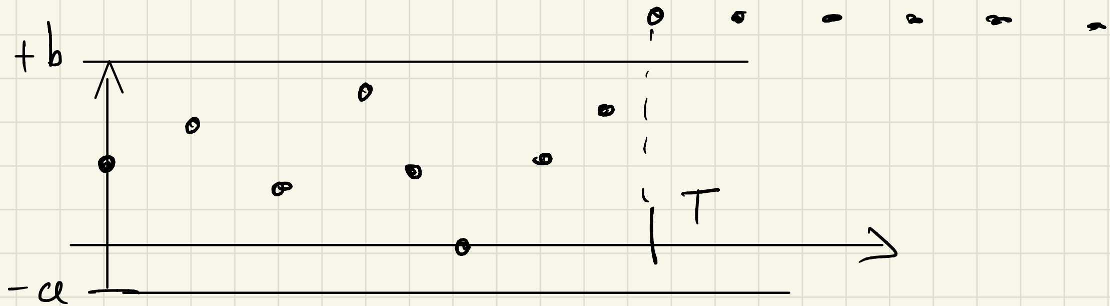
Optional stopping theorem, v3

- Let M be a martingale wrt $(\mathcal{F}_n, n \in \mathbb{N})$ be such that $|M_{n+1}(\omega) - M_n(\omega)| < C$ $\forall n \in \mathbb{N}, \forall \omega \in \Omega$ for some $C < +\infty$. (so the classical RW satisfies this)
- Let $a, b > 0$ and

$$T = \inf\{n \in \mathbb{N} : M_n \leq -a \text{ or } M_n \geq +b\}$$

(so the martingale is confined to $[-a, +b]$) "up to the time T"

Observe that $M_{T\wedge n}(\omega) \begin{cases} \geq -a-c \\ \leq +b+c \end{cases} \quad \forall n\in\mathbb{N}, \forall \omega\in\Omega$



So $\sup\limits_{n\in\mathbb{N}} \sup\limits_{\omega\in\Omega} |M_{T\wedge n}(\omega)| \leq \underbrace{\max(|a+c|, b+c)}_{<+\infty}$

So we can apply the MCT, v1, saying

(iii) $\exists\, M_{T\wedge\infty}$ s.t. $\mathbb{E}(M_{T\wedge\infty} \mid \mathcal{F}_n) = M_{T\wedge n} \quad \forall n\in\mathbb{N}$

i.e. $\mathbb{E}(M_{T\wedge\infty}) = \mathbb{E}(M_{T\wedge n}) \quad \forall n \in \mathbb{N}$

i.e. $\mathbb{E}(M_T) = \mathbb{E}(M_{T\wedge n}) \quad \forall n \in \mathbb{N}$

$$= \mathbb{E}(M_0) \qquad \text{(take } n=0\text{)}$$

<u>Rmk</u>: Here, M could be the classical RW & T is unbounded

Example

Consider $M = S$ the classical RW,

$a, b \in \mathbb{N}$, $T = \inf\{n \geq 1 : S_n \geq +b \text{ or } S_n \leq -a\}$

By the thm (v3), we have $\mathbb{E}(S_T) = \mathbb{E}(S_0) = 0$

Observation: $S_T \in \{-a, +b\}$ almost surely

$\mathbb{P}(\{T = +\infty\}) = 0$

So $\underbrace{\mathbb{E}(S_T)}_{0} = -a\,\mathbb{P}(\{S_T = -a\}) + b\,\underbrace{\mathbb{P}(\{S_T = b\})}_{1 - \mathbb{P}(\{S_T = -a\})}$

So $\mathbb{P}(\{S_T = -a\}) = \dfrac{b}{a+b}$

Question:

What about considering $M = S$ classical RW

and $T' = \inf\{n \in \mathbb{N} : S_n \geq +b\}$?

Does it hold that $\underbrace{\mathbb{E}(S_{T'})}_{b} = \underbrace{\mathbb{E}(S_0)}_{0}$? no !!

(NB: $\mathbb{P}(\{T' < +\infty\}) = 1$)

because $(M_{T' \wedge n}, n \in \mathbb{N})$ is not bounded from below !