APA Lecture 14a

Martingale convergence theorem (v2)

Let $(M_n, n \in \mathbb{N})$ be a martingale s.t.

$$\sup_{n \in \mathbb{N}} \mathbb{E}(|M_n|) < +\infty$$

Then there exists a random variable M_∞ such that $M_n \xrightarrow[n\to\infty]{} M_\infty$ a.s. That's it.

<u>Proof</u>: omitted (uses "upcrossing lemma")

Remarks about the assumption: $\sup_{n\in\mathbb{N}} \mathbb{E}(|M_n|) < +\infty$:

1°) It is stronger than $\sup_{n\in\mathbb{N}} \mathbb{E}(M_n) < +\infty$: ⊛

Actually $\mathbb{E}(M_n) = \mathbb{E}(M_0)$ by assumption,

so $\sup_{n\in\mathbb{N}} \mathbb{E}(M_n) = \sup_{n\in\mathbb{N}} \mathbb{E}(M_0) = \mathbb{E}(M_0) < +\infty$.

So ⊛ is not an assumption!

But $\mathbb{E}(|M_n|)$ is increasing in general, so the assumption on the top is a true one!

2°) If $(M_n, n \in \mathbb{N})$ is a non-negative martingale (i.e. $M_n(\omega) \geq 0 \ \forall n \in \mathbb{N}, \forall \omega \in \Omega$), then the assumption is automatically satisfied:

$$\sup_{n \in \mathbb{N}} \mathbb{E}(|M_n|) = \sup_{n \in \mathbb{N}} \mathbb{E}(M_n) = \mathbb{E}(M_0) < +\infty$$

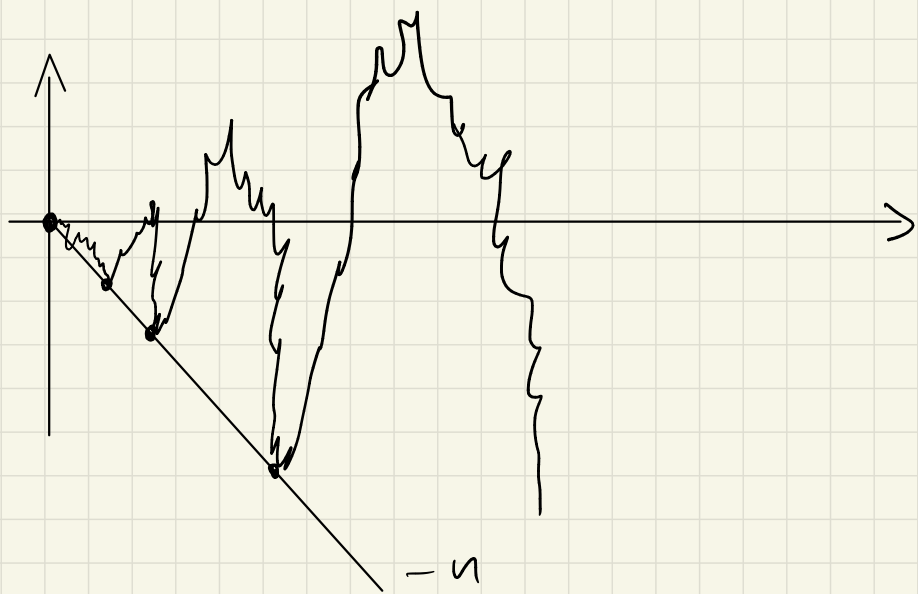
So a non-negative martingale converges a.s.

(same thing for M non-positive)

Example:

Let $(S_n, n \in \mathbb{N})$ be the classical random walk

- $M_n = S_n^2 - n$ is also a martingale



- $M_n = \exp(\alpha S_n - c_\alpha n)$ $\quad \alpha > 0$

For a given c_α, M = martingale:

$$\mathbb{E}(M_{n+1} \mid \mathcal{F}_n) = \mathbb{E}(\exp(\alpha S_{n+1} - c_\alpha(n+1)) \mid \mathcal{F}_n)$$

$$= \mathbb{E}(\exp(\alpha(S_n + X_{n+1}) - c_\alpha(n+1)) \mid \mathcal{F}_n)$$

$$= \mathbb{E}(\exp(\alpha X_{n+1})) \cdot \underbrace{\exp(\alpha S_n - c_\alpha n}_{} - c_\alpha)$$

$$= \underbrace{\frac{e^{+\alpha} + e^{-\alpha}}{2}}_{\cosh(\alpha)} \cdot \exp(-c_\alpha) \cdot M_n \underset{\substack{\uparrow \\ c_\alpha = \log(\cosh(\alpha))}}{=} M_n$$

So $M_n = \exp(\alpha S_n - \log\cosh(\alpha)\, n)$ is a martingale

<u>Application</u>: if $T_a = \inf\{n \geq 1 : |S_n| \geq a\}$

then the optional stopping thm (v3)

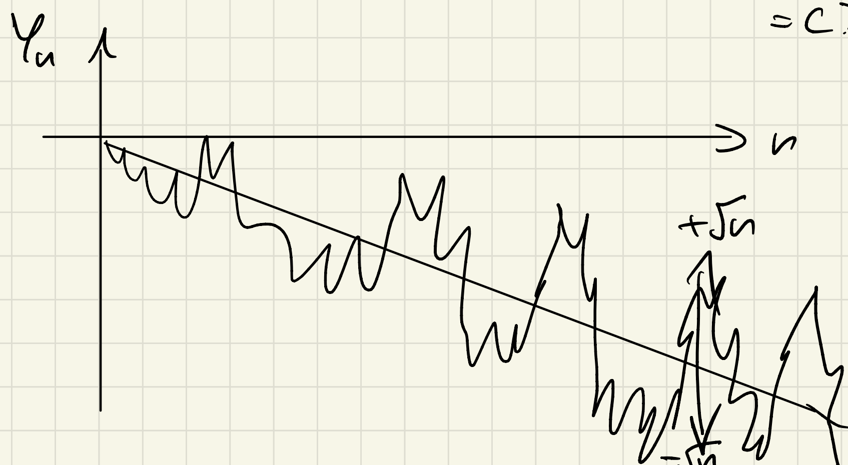
allows to deduce the value of

$$\mathbb{E}(\exp(\beta T_a)) \quad \text{for any } \beta > 0$$

and therefore the distribution of T_a.

Consider the case $\alpha = 1$: $M_n = \exp(S_n - \log\cosh(1) \cdot n)$

- M = non-negative martingale
- by the MCT (v2), $\exists\, M_\infty$ st $M_n \to M_\infty$ a.s.
- consider $Y_n = \log M_n = S_n - \underbrace{\log\cosh(1)}_{=c>0} \cdot n$



$Y_n \xrightarrow[n\to\infty]{} -\infty$ a.s.

so $\underline{M_\infty = 0}$!

So $\cdot$ $\mathbb{E}(|M_n - \underbrace{M_\infty}_{=0}|) = \mathbb{E}(|M_n|) = \mathbb{E}(M_n) =$

$= \mathbb{E}(M_0) = 1$

no L^1-convergence !

$\cdot$ $\mathbb{E}(M_\infty | \mathcal{F}_n) = \mathbb{E}(0 | \mathcal{F}_n) = 0 \neq M_n$

no closed martingale !

Puzzling fact:

$$\mathrm{Var}(M_n) = \mathbb{E}(M_n^2) - \mathbb{E}(M_n)^2$$

$$= \mathbb{E}\left(\exp(2S_n - 2\log\cosh(1)n)\right) - \mathbb{E}(M_0)^2$$

$$= \mathbb{E}(\exp(2X_1))^n \cdot \exp(-2\log\cosh(1)\cdot n) - 1^2$$

$$= \left(\frac{e^{+2}+e^{-2}}{2}\right)^n \cdot (\cosh(1))^{-2n} - 1$$

$$= \cosh(2)^n \cdot \cosh(1)^{-2n} - 1$$

$$= \Big(\underbrace{\frac{\cosh(2)}{\cosh(1)^2}}_{>1}\Big)^n - 1 \xrightarrow[n\to\infty]{} +\infty \text{ exp. fast}$$

M_n
1
n

Generalizations to sub- & supermartingales

V1: Let M be a sub- / super- martingale be such that $\sup_{n\in\mathbb{N}} \mathbb{E}(M_n^2) < +\infty$. Then there exists a r.v. M_∞ s.t.

(i) $M_n \xrightarrow[n\to\infty]{} M_\infty$ a.s.

(ii) $\mathbb{E}(|M_n - M_\infty|) \xrightarrow[n\to\infty]{} 0$

(ii) $\mathbb{E}(M_\infty \mid \mathcal{F}_n) \geq / \leq M_n \quad \forall n \in \mathbb{N}$

V2: Let $(M_n, n \in \mathbb{N})$ be a $\left.\begin{matrix}\text{sub-}\\\text{super-}\end{matrix}\right\}$ martingale

such that $\begin{cases} \sup\limits_{n\in\mathbb{N}} \mathbb{E}(M_n^+) < +\infty \\ \sup\limits_{n\in\mathbb{N}} \mathbb{E}(M_n^-) < +\infty \end{cases}$

then $\exists$ a r.v. M_∞ st $M_n \xrightarrow[n\to\infty]{} M_\infty$ a.s.

(remember $M_n^+ = \max(M_n, 0)$, $M_n^- = \max(-M_n, 0)$)

Corollary: $\left\{\begin{matrix}\text{If } M \text{ is a non-positive submartingale}\\ \text{If } M \text{ is a non-negative supermartingale}\end{matrix}\right\}$ then it converges.

Ex 1: $M_n = \exp(S_n) =$ non-negative submartingale

$$\sup_{n \in \mathbb{N}} \mathbb{E}(M_n^+) = \sup_{n \in \mathbb{N}} \mathbb{E}(M_n) = +\infty$$

so no convergence to some M_∞

Ex 2: $M_n = \exp(S_n - c'n)$ $\quad c' > \log\cosh(1)$

$\Rightarrow M_n =$ non-negative supermartingale

$\Rightarrow \exists M_\infty$ s.t. $M_n \xrightarrow[n\to\infty]{} M_\infty$ a.s.

& $M_\infty = 0$ (same reasoning as before)

If c' is large enough, then $\sup_{n \in \mathbb{N}} \mathbb{E}(M_n^2) < +\infty$

So: (ii) $\underbrace{\mathbb{E}(|M_n - \overset{0}{M_\infty}|)}_{= \mathbb{E}(|M_n|) = \mathbb{E}(M_n) \text{ is decreasing}} \xrightarrow[n\to\infty]{} 0$ ✓

(iii) $\underbrace{\mathbb{E}(\underbrace{M_\infty}_{0} \mid \mathcal{F}_n)}_{=0} \leq M_n$ ✓