APA lecture 14b

Reminder: Hoeffding's inequality

$S_n = X_1 + \dots + X_n$ iid X's with $|X_j(\omega)| \le 1$ $\forall j$ $\forall \omega$

$$\mathbb{P}(\{|S_n| \geq nt\}) \leq 2\exp\left(-n\frac{t^2}{2}\right) \quad \forall n \geq 1, \forall t > 0$$

Azuma's inequality:

Let $(M_n, n \in \mathbb{N})$ be a martingale s.t.

for $Y_n(\omega) = M_n(\omega) - M_{n-1}(\omega)$ for $n \geq 1$

we have $|Y_n(\omega)| \leq 1 \quad \forall n \geq 1, \forall \omega \in \Omega$

("finite difference martingale")

equiv: $M_n = M_0 + Y_1 + \dots + Y_n$

remember that the Y's are orthogonal (Cov $= 0$)

$$\mathbb{P}(\{|M_n - M_0| \geq nt\}) \leq 2\exp\left(-\frac{nt^2}{2}\right) \quad \forall n \geq 1, \forall t > 0$$

Proof:

$$\mathbb{P}(\{|M_n - M_0| \geq nt\}) = \mathbb{P}(\{|\sum_{i=1}^{n} Y_i| \geq nt\})$$

$$= \underbrace{\mathbb{P}(\{\sum_{i=1}^{n} Y_i \geq nt\})}_{} + \mathbb{P}(\{\sum_{i=1}^{n} Y_i \leq -nt\})$$

$$\underset{\substack{\text{Chebyshev} \\ \varphi(x) = e^{sx} \\ s \geq 0}}{\leq} \frac{\mathbb{E}(\exp(s\sum_{i=1}^{n} Y_i))}{e^{snt}}$$

$$\mathbb{E}\left(\exp\left(s\sum_{i=1}^{n} Y_i\right)\right) = \mathbb{E}\left(\mathbb{E}\left(\exp\left(s\sum_{i=1}^{n} Y_i\right) \Big| \mathcal{F}_{n-1}\right)\right)$$

$$= \mathbb{E}\left(\exp\left(s\sum_{i=1}^{n-1} Y_i\right) \cdot \mathbb{E}\left(\exp(s Y_n) \,|\, \mathcal{F}_{n-1}\right)\right)$$

Observe: $|Y_n| \leq 1$ and $\mathbb{E}(Y_n \,|\, \mathcal{F}_{n-1})$

$$= \mathbb{E}(M_n - M_{n-1} \,|\, \mathcal{F}_{n-1}) = 0$$

Remember the lemma: $\begin{cases} |Z| \leq 1 \ \ \& \ \ \mathbb{E}(Z) = 0 \\ \Longrightarrow \ \mathbb{E}(e^{sZ}) \leq e^{s^2/2} \quad \forall s \in \mathbb{R} \end{cases}$

So here: $\mathbb{E}(e^{sY_n} \,|\, \mathcal{F}_{n-1}) \leq e^{s^2/2}$

So $\mathbb{E}\left(\exp\left(s\sum_{i=1}^{n} Y_i\right)\right) \leq \mathbb{E}\left(\exp\left(s\sum_{i=1}^{n-1} Y_i\right)\right) \cdot e^{s^2/2}$

$$\leq \mathbb{E}\left(\exp\left(s\sum_{i=1}^{n-2} Y_i\right)\right) \cdot e^{s^2/2} \cdot e^{s^2/2} \leq \cdots$$

$$\leq e^{n s^2/2}$$

Finally, $\mathbb{P}(\{ M_n - M_0 \geq nt\}) \leq \dfrac{e^{ns^2/2}}{e^{snt}}$

$$= \exp\Big(-n \underbrace{\left(st - s^2/2\right)}_{\text{choose } s^* = t}\Big) \quad \forall s \geq 0$$

$\Longrightarrow \exp(-nt^2/2)$ #

Generalization

Let M be a martingale s.t.

$$Y_n(\omega) = M_n(\omega) - M_{n-1}(\omega) \in [a_n, b_n] \quad \forall n \geq 1, \forall \omega \in \Omega$$

Then $\mathbb{P}(\{|M_n - M_0| \geq nt\})$

$$\leq 2 \exp\left(-\frac{2n^2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right)$$

$\forall n \geq 1, \ \forall t > 0$

Example:

Consider $M_0 = 0$, $M_n = \sum_{i=1}^{n} \overbrace{H_i X_i}^{=Y_i}$

where the X_i are iid $\left(\mathbb{P}(\{X_i = +1\}) = \mathbb{P}(\{X_i = -1\}) = 1/2 \right)$

$\mathcal{F}_i = \sigma(X_1 \dots X_i)$ $i \geq 1$, $\mathcal{F}_0 = \{\phi, \Omega\}$

H_i are bounded & $\mathcal{F}_{i-1}$-measurable $\forall i \geq 1$

(i.e. $|H_i(\omega)| \leq K_i$)

$$\mathbb{P}(\{|M_n - \underbrace{M_0}_{0}| \geq nt\}) \leq 2 \exp\left(- \frac{n^2 t^2}{2 \sum_{j=1}^{n} K_j^2}\right)$$

Mc Diarmid's inequality

Let $n \geq 1$ be fixed and $X_1 \dots X_n$ be iid r.v.,

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ Borel-measurable and s.t.

$$|f(x_1, \dots, x_j, \dots, x_n) - f(x_1, \dots, x_j', \dots, x_n)| \leq k_j \quad \forall 1 \leq j \leq n$$

$$\forall x_1 \dots x_j, x_j' \dots x_n$$

Then $\mathbb{P}\left(\{|f(X_1, \dots X_n) - \mathbb{E}(f(X_1, \dots, X_n))| \geq nt\}\right)$

$$\leq 2 \exp\left(-\frac{n^2 t^2}{2 \sum_{j=1}^{n} k_j^2}\right)$$

Proof:

Define $\mathcal{F}_0 = \{\phi, \Omega\}$, $\mathcal{F}_j = \sigma(X_1 \dots X_j)$ $1 \leq j \leq n$

$$M_0 = \mathbb{E}(f(X_1, \dots, X_n)), \quad M_j = \mathbb{E}(f(X_1, \dots, X_n) \mid \mathcal{F}_j)$$

M is a Doob martingale: $M_n = f(X_1, \dots, X_n)$

To be checked: $|M_j - M_{j-1}| \leq k_j \quad \forall 1 \leq j \leq n$

$$|M_j - M_{j-1}| = |\underbrace{\mathbb{E}(f(X_1 \dots X_n) \mid \mathcal{F}_j)}_{= g(X_1 \dots X_j)} - \underbrace{\mathbb{E}(f(X_1 \dots X_n) \mid \mathcal{F}_{j-1})}_{= h(X_1 \dots X_{j-1})}|$$

$\leq k_j$ (see next page) #

$$g(x_1, \dots, x_j) = \mathbb{E}\big(f(x_1, \dots, x_j, X_{j+1}, \dots, X_n)\big)$$

$$h(x_1, \dots, x_{j-1}) = \mathbb{E}\big(f(x_1, \dots, x_{j-1}, X_j, X_{j+1}, \dots X_n)\big)$$

$$|g(x_1 .. x_j) - h(x_1 .. x_{j-1})| \le k_j \quad \forall (x_1 \dots x_j)$$

$$= |\mathbb{E}(f(x_1 .. x_j X_{j+1} \dots X_n)) - \mathbb{E}(f(x_1 .. x_{j-1} X_j \dots X_n))|$$

$$\le \mathbb{E}\Big(\underbrace{|f(x_1 .. x_{j-1} x_j X_{j+1} .. X_n) - f(x_1 .. x_{j-1} X_j X_{j+1} .. X_n)|}_{(\forall \omega) \le k_j}\Big)$$