Final exam

Please pay attention to the presentation of your answers! **(3 points)**

Exercise 1. Quiz. (21 points)

Answer each yes/no question below (1 pt) and provide a short justification (proof or counter-example) for your answer (2 pts).

a) Let X_1, X_2, X_3 be three random variables, such that for every $1 \leq i \leq 3$, X_i takes values in $\{-1, +1\}$ and $\mathbb{P}(\{X_i = +1\}) = \mathbb{P}(\{X_i = -1\}) = 1/2$. Is it the case that $\sigma(X_1 + X_2, X_1 + X_3) = \sigma(X_1, X_2, X_3)$?

b) Let X, Y be two random variables such that $X^2 \geq Y^2$ almost surely, $\mathbb{E}(X) = \mathbb{E}(Y)$ and $\mathbb{E}(X^2) = \mathbb{E}(Y^2)$. Does it necessarily hold that $X = Y$ almost surely ?

c) Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous cdf and $G : \mathbb{R} \to \mathbb{R}$ be the function defined as

$$G(t) = \begin{cases} 1 - F(1/t) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

Is it always the case that G is a also a cdf ?

d) Let $\phi : \mathbb{R} \to \mathbb{R}$ be the function defined as

$$\phi(t) = \begin{cases} 1 & \text{if } |t| \leq 1 \\ 2 - |t| & \text{if } 1 \leq |t| \leq 2 \\ 0 & \text{if } |t| \geq 2 \end{cases}$$

Is ϕ a characteristic function ?

Hint: Consider the 3×3 matrix $\{A_{ij} = \phi(t_i - t_j)\}_{i,j=1}^3$ with $t_1 = -1$, $t_2 = 0$ and $t_3 = +1$.

e) Do there exist two non-deterministic and i.i.d. random variables X, Y such that $X + Y$ and $2X$ have the same distribution ?

f) Let $(Y_n,\, n \in \mathbb{N})$ and $(Z_n,\, n \in \mathbb{N})$ be two sequences of random variables. Let us also define, for $n \in \mathbb{N}$:

$$\mathcal{F}_n = \sigma\left(\sum_{j=0}^{k} Y_j\, Z_{k-j},\; 0 \leq k \leq n\right)$$

Is $(\mathcal{F}_n,\, n \in \mathbb{N})$ a filtration ?

g) Let $(X_n,\, n \in \mathbb{N})$ be a sequence of random variables and $(\mathcal{F}_n^X,\, n \in \mathbb{N})$ be its natural filtration. Let also

$$T = \inf\{n \in \mathbb{N} \,:\, X_n > X_{n+1}\}$$

Is T a stopping time with respect to the filtration $(\mathcal{F}_n^X,\, n \in \mathbb{N})$?

Exercise 2. (25 points + BONUS 3 points)

Let $(Z_n,\ n \geq 1)$ be a sequence of i.i.d. $\sim \mathcal{N}(0,1)$ random variables. Let also $a \in \mathbb{R}$ and let $(X_n,\ n \in \mathbb{N})$ be the stochastic process defined recursively as

$$X_0 = 0, \quad X_{n+1} = a\, X_n + Z_{n+1}, \quad n \geq 0$$

Moreover, let $(Y_n,\ n \geq 1)$ be the sequence of random variables defined as

$$Y_n = \sum_{j=0}^{n-1} X_j\, X_{j+1}, \quad n \geq 1$$

a) Compute $\mathbb{E}(Y_n)$ for $n \geq 1$, and when $-1 < a < +1$, compute $\lim_{n\to\infty} \mathbb{E}(Y_n/n)$.

From now on, we assume that $\boldsymbol{a} = \mathbf{0}$.

b) Show that

$$\frac{Y_n}{n} \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$$

c) Show that for $0 \leq s \leq \frac{1}{2}$, it holds that

$$\mathbb{E}(\exp(sY_n)) \leq \frac{1}{\left(\sqrt{1-2s^2}\right)^n}$$

Hint: Condition successively on $\mathcal{F}_j = \sigma(X_1, \ldots, X_j)$, $j = n-1, n-2, \ldots, 1$ and use the inequalities:

$$\mathbb{E}(\exp(szX)) \leq \mathbb{E}(\exp(szX + s^2X^2)) \overset{(*)}{\leq} \frac{1}{\sqrt{1-2s^2}} \exp(s^2z^2)$$

valid for $0 \leq s \leq \frac{1}{2}$, $z \in \mathbb{R}$ and $X \sim \mathcal{N}(0,1)$. Please note that the first inequality is obvious; it is useful in the first step of the computation.

BONUS: Prove the last inequality $(*)$.

d) Deduce from there that for every $t > 0$, there exists a constant $C > 0$ (possibly depending on t) such that for every $n \geq 1$,

$$\mathbb{P}(\{Y_n \geq nt\}) \leq \exp(-Cn)$$

Hint: In order to simplify computations, you may use the inequality $-\log(1-x) \leq 2x$, valid for $0 \leq x \leq \frac{1}{2}$.

e) Is the process $(Y_n,\ n \geq 1)$ a martingale with respect to $(\mathcal{F}_n,\ n \geq 1)$? Justify.

Exercise 3. (25 points)

Let $0 < \sigma < 1$ and $(Z_n,\, n \in \mathbb{N})$ be a collection of i.i.d. and zero-mean random variables such that $\mathrm{Var}(Z_1) = \sigma^2$ and $|Z_n(\omega)| \leq 1$ for all $n \in \mathbb{N}$ and $\omega \in \Omega$. Let also X, Y be the two stochastic processes defined recursively as

$$\begin{cases} X_0 = 1, \quad X_{n+1} = X_n\,(1 + Z_{n+1}), \quad n \geq 0 \\ Y_0 = 1, \quad Y_{n+1} = Y_n\,(1 - Z_{n+1}), \quad n \geq 0 \end{cases}$$

a) Compute recursively $\mathrm{Cov}(X_n, Y_n)$ for $n \in \mathbb{N}$. Is this covariance increasing or decreasing with n ?

Now, let $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$ for $n \geq 1$, and

$$M_n = X_n + Y_n, \quad n \geq 0$$

b) Does it hold that $(M_n,\, n \in \mathbb{N})$ is a Markov process with respect to the filtration $(\mathcal{F}_n,\, n \in \mathbb{N})$, i.e. that

$$\mathbb{E}(g(M_{n+1}) \,|\, \mathcal{F}_n) = \mathbb{E}(g(M_{n+1}) \,|\, M_n)$$

for every $n \geq 0$ and $g \in C_b(\mathbb{R})$? Justify.

c) Show that the process $(M_n,\, n \in \mathbb{N})$ is a martingale with respect to the filtration $(\mathcal{F}_n,\, n \in \mathbb{N})$.

d) Compute the process $(A_n,\, n \in \mathbb{N})$ defined recursively as

$$A_0 = 0, \quad A_{n+1} = A_n + \mathbb{E}(M_{n+1}^2 \,|\, \mathcal{F}_n) - M_n^2, \quad n \geq 0$$

e) What do you know about the process $(M_n^2 - A_n,\, n \in \mathbb{N})$?

f) Does there exist a random variable M_∞ such that $M_n \underset{n\to\infty}{\to} M_\infty$ almost surely ? Justify.

g) Does there exist a random variable M_∞ such that $M_n \overset{L^2}{\underset{n\to\infty}{\to}} M_\infty$? Justify.

Exercise 4. (16 points)

Let $M = (M_n,\, n \in \mathbb{N})$ be a stochastic process defined recursively as follows:

$$M_0 = x < 0, \quad M_{n+1} = \begin{cases} \dfrac{3M_n + 1}{2} & \text{with probability } \frac{1}{2} \\[2ex] \dfrac{M_n}{2} & \text{with probability } \frac{1}{2} \end{cases}$$

a) Show that the process $(M_n,\, n \in \mathbb{N})$ is submartingale.

Let us now consider the stopping time $T = \inf\{n \geq 1 \,:\, M_n \geq 0\}$, as well as the stopped submartingale $N = M^T$ defined as

$$N_n = M_n^T = M_{T \wedge n} = M_{\min(T,n)} \quad \text{for } n \in \mathbb{N}$$

b) Explain why there exists a random variable N_∞ such that $N_n \underset{n\to\infty}{\to} N_\infty$ almost surely.

c) Does it hold that $\mathbb{E}(N_\infty | \mathcal{F}_n) = N_n$ for every $n \in \mathbb{N}$? Justify.

d) To what interval in $\mathbb{R}$ does the random variable N_∞ belong ?

Remark: Of course, $\mathbb{R}$ itself is a valid answer to the previous question, but we are actually asking here for the interval of minimal size to which N_∞ is guaranteed to belong.