Differential Geometry II - Smooth Manifolds
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Lecturer: Dr. N. Tsakanikas
Assistant: L. E. Rösler

The Projective Space

So far, most of the smooth manifolds we encountered in this course were intrinsically subspaces of some Euclidean space $\mathbb{R}^n$. However, the set-up of the general theory (that is, endowing topological manifolds with a smooth structure) is designed precisely to allow our objects of study to come along as abstract spaces, rather than requiring them to be subsets of $\mathbb{R}^n$. So it would be nice to see an example of a smooth manifold which takes advantage of this abstract set-up. An elementary yet important example is the real projective space $\mathbb{RP}^n$, which will be described in this short note.

The underlying set of $\mathbb{RP}^n$:

Let $n \in \mathbb{N}^*$. There is a natural group action of $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$ on $\mathbb{R}^{n+1} \setminus \{0\}$ given by

$$\begin{aligned} \mathbb{R}^\times \times (\mathbb{R}^{n+1} \setminus \{0\}) &\to \mathbb{R}^{n+1} \setminus \{0\} \\ (\lambda, x) &\mapsto \lambda x. \end{aligned}$$

As with any group action, we can form the quotient set, whose points are the orbits of the action. Concretely, we define the *real projective space of dimension* n, denoted by $\mathbb{RP}^n$, to be the quotient of the above action, i.e.,

$$\mathbb{RP}^n := \left(\mathbb{R}^{n+1} \setminus \{0\}\right) /_{\mathbb{R}^\times}.$$

Note that $\mathbb{RP}^n$ comes equipped with a natural surjection

$$\begin{aligned} \pi \colon \mathbb{R}^{n+1} \setminus \{0\} &\to \mathbb{RP}^n \\ x &\mapsto [x] := \mathbb{R}^\times \cdot x. \end{aligned}$$

In particular, notice that points of $\mathbb{RP}^n$ are in one-to-one correspondence with one-dimensional subspaces of $\mathbb{R}^{n+1}$: if $[x] \in \mathbb{RP}^n$, then $[x] \cup \{0\} = \mathbb{R} \cdot x$ is the one-dimensional subspace of $\mathbb{R}^{n+1}$ generated by x, while if L is any one-dimensional subspace of $\mathbb{R}^{n+1}$, then $L \setminus \{0\} = [x]$ for any $x \in L \setminus \{0\}$. (This is the geometric picture you should have in mind when thinking about $\mathbb{RP}^n$.) If

$$x = (x_0, \ldots, x_n)$$

is a point of $\mathbb{R}^{n+1} \setminus \{0\}$, then we denote by

$$\pi(x) = [x] = [x_0 : \ldots : x_n]$$

the corresponding point of $\mathbb{RP}^n$. Note that $[x_0 : \ldots : x_n] = [y_0 : \ldots : y_n]$ if and only if there exists $\lambda \neq 0$ such that $\lambda x_i = y_i$ for all i.

The topology of $\mathbb{RP}^n$:

By definition, $\mathbb{RP}^n$ is a quotient of $\mathbb{R}^{n+1} \setminus \{0\}$, and the latter can be equipped with its natural Euclidean topology. Recall that in general there is a procedure with which the quotient of some topological space can be equipped with a natural topology. Concretely, one can easily show that the collection

$$\mathcal{T}_{\mathbb{RP}^n} := \big\{U \subseteq \mathbb{RP}^n \mid \pi^{-1}(U) \subseteq \mathbb{R}^{n+1} \setminus \{0\} \text{ is open}\big\}$$

is a topology on $\mathbb{RP}^n$. Moreover, if we endow $\mathbb{RP}^n$ with this topology, then the quotient map $\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is continuous, and a map $f\colon \mathbb{RP}^n \to X$ from $\mathbb{RP}^n$ to some topological space X is continuous if and only if so is the composite map $f \circ \pi$. The same is true for any subset $A \subseteq \mathbb{RP}^n$ endowed with the subspace topology. (If this is new for you, you can verify this as an exercise.)

At this point, there are several things that need to be checked about the topological space $\mathbb{RP}^n$.

Exercise 1: Show that $\mathbb{RP}^n$ is *Hausdorff* by going through the following steps:

(i) Show that the quotient map $\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is open.

(ii) Show that the set

$$\widetilde{\Delta} := \big\{(x, y) \in (\mathbb{R}^{n+1} \setminus \{0\}) \times (\mathbb{R}^{n+1} \setminus \{0\}) \mid [x] = [y]\big\}$$

is closed in $(\mathbb{R}^{n+1} \setminus \{0\}) \times (\mathbb{R}^{n+1} \setminus \{0\})$.

(iii) Show that the set

$$\Delta := \big\{\big([x], [x]\big) \in \mathbb{RP}^n \times \mathbb{RP}^n \mid [x] \in \mathbb{RP}^n\big\}$$

is closed in $\mathbb{RP}^n \times \mathbb{RP}^n$.

(iv) Conclude that $\mathbb{RP}^n$ is Hausdorff.

[Hint: Use (iii) and that the collection

$$\big\{U \times V \mid U, V \in \mathcal{T}_{\mathbb{RP}^n}\big\}$$

is a basis for the topology of $\mathbb{RP}^n \times \mathbb{RP}^n$ by definition of the product topology.]

Exercise 2: Show that $\mathbb{RP}^n$ is *second-countable.*

[Hint: Use *Exercise* 1(i).]

Exercise 3: Show that $\mathbb{RP}^n$ is *locally Euclidean of dimension* n as follows.

(i) For each $0 \leq i \leq n$, set

$$U_i := \big\{[x_0 : \ldots : x_n] \mid x_i \neq 0\big\} \subseteq \mathbb{RP}^n.$$

Show that U_i is open, and that

$$\mathbb{RP}^n = \bigcup_{i=0}^{n} U_i.$$

(ii) For each $0 \leq i \leq n$, consider the map

$$\begin{aligned} \varphi_i \colon U_i &\to \mathbb{R}^n \\ [x_0 : \ldots : x_n] &\mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i} \right). \end{aligned}$$

Show first that φ_i is well-defined, and then that it is a homeomorphism. Conclude that $\mathbb{RP}^n$ is locally Euclidean of dimension n.

Exercise 4:

(i) Show that $\mathbb{RP}^n$ is connected.

(ii) Show that the restriction of π to $S^n \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ is still surjective. Conclude that $\mathbb{RP}^n$ is compact.

By the above exercises we infer that $\mathbb{RP}^n$ is an *n-dimensional topological manifold*, which is additionally compact and connected.

Before continuing the study of $\mathbb{RP}^n$, a few words about the open subsets U_i defined in *Exercise* 3(i) are in order. The open cover $\mathbb{RP}^n = \bigcup_{i=0}^n U_i$ is called the *standard open cover* of $\mathbb{RP}^n$. The equality, for example, $\varphi_n([x]) = y$, means that the line corresponding to $[x]$ meets the plane $\mathbb{R}^n \times \{1\}$ at the point $(y, 1)$. The complement of U_n consists of those lines which do not intersect the plane $\mathbb{R}^n \times \{1\}$, which (as you may convince yourself) are precisely the lines contained in $\mathbb{R}^n \times \{0\}$. Hence, we may somewhat suggestively write

$$\mathbb{RP}^n = U_n \sqcup \mathbb{P}\big(\mathbb{R}^n \times \{0\}\big) \cong \mathbb{R}^n \sqcup \mathbb{RP}^{n-1}.$$

We may thus regard $\mathbb{RP}^n$ as a compactification of $\mathbb{R}^n$ by adding the points of $\mathbb{RP}^{n-1}$, which from this point of view are often called *points at infinity.* In particular, the real projective line $\mathbb{RP}^1$ ($n = 1$) may be regarded a one-point compactification of the real line $\mathbb{R}^1$, obtained by adding to it a "point at infinity", and the real projective plane $\mathbb{RP}^2$ ($n = 2$) may be viewed as a compactification of the real plane $\mathbb{R}^2$ by adding to it a "line at infinity".

The smooth structure of $\mathbb{RP}^n$:

The standard open cover

$$\mathbb{RP}^n = \bigcup_{i=0}^n U_i.$$

together with the homeomorphisms

$$\varphi_i \colon U_i \to \mathbb{R}^n, \ [x_0 : \ldots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i} \right), \ 0 \leq i \leq n$$

determine an atlas of $\mathbb{RP}^n$. According to part (a) of *Exercise* 1*, Sheet* 2, to obtain a smooth structure on $\mathbb{RP}^n$, it only remains to check that the charts $\{(U_i, \varphi_i)\}_{0 \leq i \leq n}$ are smoothly compatible.

Exercise 5: Let $0 \leq i < j \leq n$. Show that the transition map from (U_i, φ_i) to (U_j, φ_j) is a diffeomorphism by computing that

$$\begin{aligned}
\varphi_j \circ \varphi_i^{-1} \colon \mathbb{R}^n_{x_j \neq 0} &\to \mathbb{R}^n_{x_{i+1} \neq 0} \\
(x_1, \ldots, x_n) &\mapsto \frac{1}{x_j} \left(x_1, \ldots, x_i, 1, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n \right),
\end{aligned}$$

and

$$\begin{aligned}
\varphi_i \circ \varphi_j^{-1} \colon \mathbb{R}^n_{x_{i+1} \neq 0} &\to \mathbb{R}^n_{x_j \neq 0} \\
(x_1, \ldots, x_n) &\mapsto \frac{1}{x_{i+1}} \left(x_1, \ldots, x_i, x_{i+2}, \ldots, x_j, 1, x_{j+1}, \ldots, x_n \right).
\end{aligned}$$

It follows from *Exercise* 5 that

$$\mathcal{A}_{\mathbb{RP}^n} := \big\{ (U_i, \varphi_i) \big\}_{i=0}^n$$

is a smooth atlas for $\mathbb{RP}^n$, and the smooth structure it induces is referred to as the *standard* one. Thus, we now have a smooth manifold, namely $\mathbb{RP}^n$, which is not intrinsically defined as a subset of $\mathbb{R}^n$!

Remark. A posteriori, the so-called *Whitney's embedding theorem* asserts that there is a smooth embedding $\mathbb{RP}^n \hookrightarrow \mathbb{R}^{2n}$ (and the $2n$ is in fact minimal if n is a power of 2), so in principle we can also realize the smooth manifold $\mathbb{RP}^n$ as a submanifold of $\mathbb{R}^{2n}$. But it would be very awkward if we were only able to speak about $\mathbb{RP}^n$ as a smooth manifold once we find such an embedding, so the flexibility of defining it abstractly is certainly very helpful.

Further exercises about $\mathbb{RP}^n$:

Exercise 6: Prove the following assertions:

(i) The quotient map $\pi \colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is smooth.

(ii) A map $F \colon \mathbb{RP}^n \to M$ to a smooth manifold M is smooth if and only if the composite map $F \circ \pi \colon \mathbb{R}^{n+1} \setminus \{0\} \to M$ is smooth.

Exercise 7: Show that $\mathbb{RP}^1 \cong \mathbb{S}^1$ as smooth manifolds.

[Hint: To define an appropriate map, it might be helpful to use the identifications $\mathbb{R}^2 \cong \mathbb{C}$ and $\mathbb{S}^1 \cong \{ z \in \mathbb{C} \mid |z| = 1 \}$.]

Exercise 8: Show that the quotient map $\pi \colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is a smooth submersion, and that the kernel of the differential $d\pi_p \colon T_p\big(\mathbb{R}^{n+1} \setminus \{0\}\big) \to T_{[p]}\mathbb{RP}^n$ is the subspace generated by p.

Exercise 9: Let $P \colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{k+1} \setminus \{0\}$ be a smooth map, and suppose that for some $d \in \mathbb{Z}$ we have $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R}^\times$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Show that the map $\widetilde{P} \colon \mathbb{RP}^n \to \mathbb{RP}^k$ given by $\widetilde{P}\big([x]\big) = [P(x)]$ is well-defined and smooth.

Exercise 10: Let V be a real vector space of dimension $n+1$. Define $\mathbb{P}(V)$ analogously to $\mathbb{RP}^n$, and construct a smooth structure on it. Show that the smooth structure is independent of the choice of a basis of V.

Exercise 11: Show that the map

$$F\colon \mathbb{R}^n \to \mathbb{RP}^n,\ (x^1, \ldots, x^n) \mapsto [x^1 : \cdots : x^n : 1]$$

is a diffeomorphism onto a dense open subset of $\mathbb{RP}^n$.

Exercise 12: Consider the smooth map

$$F\colon \mathbb{R}^2 \to \mathbb{RP}^2,\ (x, y) \mapsto [x : y : 1]$$

and the smooth vector field X on $\mathbb{R}^2$ defined by

$$X = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y}.$$

Show that there is a smooth vector field Y on $\mathbb{RP}^2$ that is F-related to X, and compute its coordinate representation in terms of each of the charts defined in *Exercise* 3(ii).