1 The Dual of a Vector Space

Let V be a finite-dimensional $\mathbb{R}$-vector space. A *covector* on V is a real-valued linear functional on V, that is, a linear map $\omega\colon V \to \mathbb{R}$. It is straightforward to check that the set of all covectors on V is an $\mathbb{R}$-vector space under the obvious operations of pointwise addition and scalar multiplication. It is denoted by V^* and is called *the dual space of* V. The next proposition expresses the most important fact about V^*.

Proposition 1. *Let V be an $\mathbb{R}$-vector space of dimension n. Given any basis $(E_1, \dots, E_n)$ for V, consider the covectors $\varepsilon^1, \dots, \varepsilon^n \in V^*$ defined by*

$$\varepsilon^i(E_j) = \delta^i_j.$$

Then $(\varepsilon^1, \dots, \varepsilon^n)$ is a basis for V^*, called the dual basis to (E_j). *In particular,*

$$\dim_{\mathbb{R}} V = \dim_{\mathbb{R}} V^*.$$

In general, if (E_j) is a basis for V and if (ε^i) is its dual basis, then for any vector $v = v^j E_j \in V$ we have

$$\varepsilon^i(v) = v^j \varepsilon^i(E_j) = v^j \delta^i_j = v^i.$$

Thus, the i-th basis covector ε^i picks out the i-th component of a vector with respect to the basis (E_j).

More generally, we can express an arbitrary covector $\omega \in V^*$ in terms of the dual basis as

$$\omega = \omega_i \varepsilon^i,$$

where the i-th component is determined by $\omega_i = \omega(E_i)$. Thus, the action of the given covector $\omega \in V^*$ on a vector $v = v^j E_j \in V$ is

$$\omega(v) = \omega_i v^j \varepsilon^i(E_j) = \omega_i v^i.$$

Let V and W be $\mathbb{R}$-vector spaces and let $A\colon V \to W$ be a linear map. *The dual map of A* is the linear map $A^*\colon W^* \to V^*$ defined by

$$(A^*\omega)(v) \coloneqq \omega(Av), \ \ \omega \in W^*, \ \ v \in V.$$

Proposition 2. *The dual map satisfies the following properties:*

(a) $(A \circ B)^* = B^* \circ A^*$.

(b) $(\mathrm{Id}_V)^* = \mathrm{Id}_{V^*}$.

Therefore, the assignment that sends a vector space to its dual space and a linear map to its dual linear map is a contravariant functor from the category of $\mathbb{R}$-vector spaces to itself.