Differential Geometry II - Smooth Manifolds
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Multilinear Algebra

Contents

1 The Dual of a Vector Space

Definition 1. Let V be a finite-dimensional real vector space.

(a) A *covector* on V is a real-valued linear functional on V, i.e., a linear map $\omega \colon V \to \mathbb{R}$.

(b) The set of all covectors on V is a real vector space under the obvious operations of pointwise addition and scalar multiplication. It is denoted by V^* and is called *the dual space of* V.

The next proposition expresses the most important fact about V^*.

Proposition 2. *Let V be a real vector space of dimension n. Given any basis $(E_1, \ldots, E_n)$ for V, consider the covectors $\varepsilon^1, \ldots, \varepsilon^n \in V^*$ defined by*

$$\varepsilon^i(E_j) = \delta^i_j.$$

Then $(\varepsilon^1, \ldots, \varepsilon^n)$ is a basis for V^*, called the dual basis to (E_j). *In particular,*

$$\dim_{\mathbb{R}} V = \dim_{\mathbb{R}} V^*.$$

Proof. Exercise! □

In general, if (E_j) is a basis for V and if (ε^i) is its dual basis, then for any vector $v = v^j E_j \in V$ we have

$$\varepsilon^i(v) = v^j \varepsilon^i(E_j) = v^j \delta^i_j = v^i.$$

Thus, the i-th basis covector ε^i picks out the i-th component of a vector with respect to the basis (E_j).

More generally, we can express an arbitrary covector $\omega \in V^*$ in terms of the dual basis as

$$\omega = \omega_i \varepsilon^i,$$

where the i-th component is determined by $\omega_i = \omega(E_i)$. Thus, the action of the given covector $\omega \in V^*$ on a vector $v = v^j E_j \in V$ is

$$\omega(v) = \omega_i v^j \varepsilon^i(E_j) = \omega_i v^i.$$

Let V and W be real vector spaces and let $A \colon V \to W$ be a linear map. *The dual map of A* is the linear map $A^* \colon W^* \to V^*$ defined by

$$(A^*\omega)(v) \coloneqq \omega(Av), \ \omega \in W^*, \ v \in V.$$

It is straightforward to check that it satisfies the following properties:

(a) $(A \circ B)^* = B^* \circ A^*$.

(b) $(\mathrm{Id}_V)^* = \mathrm{Id}_{V^*}$.

Proposition 3. *The assignment that sends a vector space to its dual space and a linear map to its dual linear map is a contravariant functor from the category of real vector spaces to itself.*

Another important fact about the dual of a finite-dimensional vector space is the following.

Proposition 4. *Let V be a finite-dimensional real vector space. For any given $v \in V$, define a linear functional $\xi(v)$ by*

$$\begin{aligned} \xi(v) \colon V^* &\to \mathbb{R} \\ \omega &\mapsto \xi(v)(\omega) := \omega(v). \end{aligned}$$

Then $\xi(v) \in (V^*)^*$, that is, $\xi(v)$ is a linear functional on V^*. Moreover, the map

$$\begin{aligned} \xi \colon V &\to (V^*)^* \\ v &\mapsto \xi(v) \end{aligned}$$

is an $\mathbb{R}$-linear isomorphism, which is canonical *(it is defined without reference to any basis).*

Proof. The proof that both $\xi(v)$ and ξ are linear maps are left as exercises. Since by Proposition 2 we have

$$\dim V = \dim V^* = \dim (V^*)^*,$$

it suffices to prove that ξ is injective. To this end, let $v \in V$ be non-zero, complete it to a basis $v = E_1, E_2, \ldots, E_n$ of V, and let (ε^i) be its dual basis. Then

$$\xi(v)(\varepsilon^1) = \varepsilon^1(v) = \varepsilon^1(E_1) = 1,$$

so $\xi(v) \neq 0$. Therefore, $\ker \xi = 0$; in other words, ξ is injective, as desired. □

Due to Proposition 4, the real number $\omega(v)$ obtained by applying a covector ω to a vector v is sometimes denoted by either of the more symmetric-looking notations $\langle \omega, v \rangle$ or $\langle v, \omega \rangle$; both expressions can be thought of either as the action of the covector $\omega \in V^*$ on the vector $v \in V$, or as the action of the linear functional $\xi(v) \in V^{**}$ on the element $\omega \in V^*$. There should be no cause for confusion with the use of the same angle bracket notation for inner products: whenever one of the arguments is a vector and the other a covector, the notation $\langle \omega, v \rangle$ is always to be interpreted as the natural pairing between vectors and covectors, not as an inner product.

There is also a symmetry between bases and dual bases for a finite-dimensional vector space V: any basis for V determines a dual basis for V^*, and conversely, any basis for V^* determines a dual basis for $V^{**} = V$. If (ε^i) is the basis for V^* dual to a basis (E_j) for V, then (E_j) is the basis dual to (ε^i), because both statements are equivalent to the relation $\langle \varepsilon^i, E_j \rangle = \delta^i_j$.

2 Multilinear Maps and Tensors

In the preceding section, we defined and briefly examined the dual of a vector space (in the finite-dimensional case), which is the space of real-valued linear functions on the given vector space. A natural, and from the point of view of (differential) geometry very important, generalization is to consider functions with several arguments, which are linear in each individual argument. These are called *multilinear* functions.

Definition 5. Let $V_1, \dots, V_k$ and W be real vector spaces. A map $F\colon V_1 \times \cdots \times V_k \to W$ is called *multilinear* if it is linear as a function of each variable separately when the others are held fixed; that is, if $1 \le i \le k$ is arbitrary, and if we are given elements $v_i, v_i' \in V_i$ and real numbers $a, a' \in \mathbb{R}$, then

$$F(v_1, \dots, av_i + a'v_i', \dots, v_k) = aF(v_1, \dots, v_i, \dots, v_k) + a'F(v_1, \dots, v_i', \dots, v_k).$$

Denote by $L(V_1, \dots, V_k; W)$ the set of multilinear maps from $V_1 \times \cdots \times V_k$ to W, and note that $L(V_1, \dots, V_k; W)$ has the structure of a real vector space. In the special case when $V_1 = \ldots = V_k = V$ and $W = \mathbb{R}$, we often call an element of the space $L(V, \dots, V; \mathbb{R})$ a *k-multilinear function* on V; see Definition 10.

Now, if the target space is $W = \mathbb{R}$, then there is a simple operation with which one can succesively build multilinear maps.

Definition 6. Let $V_1, \dots, V_k$ and $W_1, \dots, W_l$ be real vector spaces, and consider $F \in L(V_1, \dots, V_k; \mathbb{R})$ and $G \in L(W_1, \dots, W_l; \mathbb{R})$. The function

$$\begin{aligned} F \otimes G\colon V_1 \times \cdots \times V_k \times W_1 \times \cdots \times W_l &\to \mathbb{R} \\ (v_1, \dots, v_k, w_1, \dots, w_l) &\mapsto F(v_1, \dots, v_k)\, G(w_1, \dots, w_l) \end{aligned}$$

is called the *tensor product of F and G*.

Exercise 7.

(a) Show that, given F and G as above, the function $F \otimes G$ is multilinear, that is,

$$F \otimes G \in L(V_1, \dots, V_k, W_1, \dots, W_l; \mathbb{R}).$$

(b) Show that the tensor product operation

$$\begin{aligned} - \otimes -\colon L(V_1, \dots, V_k; \mathbb{R}) \times L(W_1, \dots, W_l; \mathbb{R}) &\to L(V_1, \dots, V_k, W_1, \dots, W_l; \mathbb{R}) \\ (F, G) &\mapsto F \otimes G \end{aligned}$$

is *bilinear*, i.e., multilinear with two variables, and *associative*, i.e., for any multilinear real-valued functions F, G, H, we have $F \otimes (G \otimes H) = (F \otimes G) \otimes H$.

Given a finite-dimensional real vector space V, we described in section 1 how to obtain a basis for the dual space $V^* = L(V; \mathbb{R})$ from a basis for V. With the above operation at hand, we may now generalize this to the space $L(V_1, \dots, V_k; \mathbb{R})$.

Proposition 8. *Let $V_1, \dots, V_k$ be $\mathbb{R}$-vector spaces of dimensions $n_1, \dots n_k$, respectively. For each $1 \le j \le k$, let $\left(E_1^{(j)}, \dots, E_{n_j}^{(j)}\right)$ be a basis of V_j, and denote by $\left(\varepsilon_{(j)}^1, \dots, \varepsilon_{(j)}^{n_j}\right)$ the corresponding dual basis of V_j^*. Then the set*

$$\mathcal{B} := \left\{ \varepsilon_{(1)}^{i_1} \otimes \cdots \otimes \varepsilon_{(k)}^{i_k} \mid 1 \le i_1 \le n_1, \dots, 1 \le i_k \le n_k \right\}$$

is a basis for $L(V_1, \dots, V_k; \mathbb{R})$, which therefore has dimension $n_1 \dots n_k$.

Proof. First, given $F \in L(V_1, \dots, V_k; \mathbb{R})$, define for each multi-index $I = (i_1, \dots, i_k)$ with $1 \le i_j \le n_j$ for all $1 \le j \le k$, a number $F_I \in \mathbb{R}$ by

$$F_I := F\left(E_{i_1}^{(1)}, \dots, E_{i_k}^{(k)}\right).$$

Also, use the short-hand notation

$$\varepsilon^{\otimes I} := \varepsilon_{(1)}^{i_1} \otimes \cdots \otimes \varepsilon_{(k)}^{i_k}.$$

We will show that

$$F = \sum_I F_I \, \varepsilon^{\otimes I},$$

where the sum is taken over all multi-indices as above, and thereby show that $\mathcal{B}$ spans $L(V_1, \dots, V_k; \mathbb{R})$. To this end, take $(v_1, \dots, v_k) \in V_1 \times \cdots \times V_k$. For integers i_j between 1 and n_j, let $v_j^{i_j} \in \mathbb{R}$ be the coefficient of v_j with respect to the basis $\left(E_1^{(j)}, \dots, E_{n_j}^{(j)}\right)$, i.e.,

$$v_j^{i_j} = \varepsilon_{(j)}^{i_j}(v_j).$$

Then by the multilinearity of F we have

$$F(v_1, \dots, v_k) = \sum_I v_1^{i_1} \cdots v_k^{i_k} F\left(E_{i_1}^{(1)}, \dots, E_{i_k}^{(k)}\right) = \sum_I v_1^{i_1} \cdots v_k^{i_k} F_I.$$

On the other hand, we have

$$\left[\sum_I F_I \, \varepsilon^{\otimes I}\right](v_1, \dots, v_k) = \sum_I F_I \, \varepsilon^{\otimes I}(v_1, \dots, v_k) = \sum_I v_1^{i_1} \cdots v_k^{i_k} F_I.$$

Hence F and $\sum_I F_I \varepsilon^{\otimes I}$ agree at any k-tuple and thus are equal, so $\mathcal{B}$ indeed spans $L(V_1, \dots, V_k; \mathbb{R})$.

Finally, in order to see that $\mathcal{B}$ is linearly independent, suppose that we have

$$\sum_I \lambda_I \, \varepsilon^{\otimes I} = 0$$

for some real numbers $\lambda_I \in \mathbb{R}$ indexed by multi-indices I. Evaluating both sides at $\left(E_{i_1}^{(1)}, \dots, E_{i_k}^{(k)}\right)$ for some fixed multi-index $I = (i_1, \dots, i_k)$, we obtain by the same computation as above that $\lambda_I = 0$. Hence, $\mathcal{B}$ is linearly independent. $\square$

The proof of Proposition 8 shows also that the components $F_{i_1\ldots i_k}$ of a multilinear function F in terms of the basis elements in $\mathcal{B}$ are given by

$$F_{i_1\ldots i_k} = F\left(E^{(1)}_{i_1}, \ldots, E^{(k)}_{i_k}\right).$$

Thus, F is completely determined by its action on all possible sequences of basis vectors.

Remark 9. You might have already encountered the abstract construction of the *tensor product of vector spaces*. If so, then regarding the above discussion (which shows that the real vector space $L(V_1, \ldots, V_k; \mathbb{R})$ can be viewed as the set of all linear combinations of objects of the form $\omega^1 \otimes \cdots \otimes \omega^k$, where $\omega^i \in V_i^*$ are covectors), one should remark the following: given finite-dimensional real vector spaces $V_1, \ldots, V_k$, there is a canonical isomorphism

$$V_1^* \otimes \cdots \otimes V_k^* \cong L(V_1, \ldots, V_k; \mathbb{R}),$$

which is induced by the multilinear map

$$\begin{aligned} \Phi\colon V_1^* \times \ldots \times V_k^* &\to L(V_1, \ldots, V_k; \mathbb{R}) \\ \Phi\left(\omega^1, \ldots, \omega^k\right)(v_1, \ldots, v_k) &\coloneqq \left(\omega^1 \otimes \cdots \otimes \omega^k\right)(v_1, \ldots, v_k) \\ &= \omega^1(v_1) \cdots \omega^k(v_k). \end{aligned}$$

Under this canonical isomorphism, abstract tensors correspond to the concrete tensor product of multilinear functions defined above. As it is a natural isomorphism, we may use the expression $V_1^* \otimes \cdots \otimes V_k^*$ as a notation for $L(V_1, \ldots, V_k; \mathbb{R})$ (this is a typical example of slight abuse of notation, where one identifies naturally isomorphic objects). Finally, using Proposition 4, we also obtain a canonical identification

$$V_1 \otimes \cdots \otimes V_k \cong L(V_1^*, \ldots, V_k^*; \mathbb{R}).$$

Therefore, we may view the above construction as a concrete construction of the abstract tensor product.

Let us now turn our attention to various spaces of multilinear functions on a finite-dimensional real vector space that naturally appear in (differential) geometry.

Definition 10. Let V be a finite-dimensional real vector space. For any integer $k \geq 1$, we denote by $T^k(V^*)$ the space of k-multilinear functions on V, i.e.,

$$T^k(V^*) \coloneqq L(\underbrace{V, \ldots, V}_{k \text{ times}}; \mathbb{R}) \cong \underbrace{V^* \otimes \ldots \otimes V^*}_{k \text{ copies}}.$$

By convention, we also define $T^0(V^*) \coloneqq \mathbb{R}$. The elements of $T^k(V^*)$ are often referred to as *covariant k-tensors on V*.

Observe that every linear functional $\omega\colon V \to \mathbb{R}$ is (trivially) multilinear, so a covariant 1-tensor is just a covector on V. Thus,

$$T^1(V^*) = V^*.$$

According to Proposition 8, we obtain a basis for $T^k(V^*)$ as follows. Assume that V has dimension n, let $(E_1, \dots, E_n)$ be a basis for V and denote by $(\varepsilon^1, \dots, \varepsilon^n)$ the dual basis for V^*. For a multi-index $I = (i_1, \dots, i_k)$, where $1 \leq i_j \leq n$ for all j, define the *elementary covariant k-tensor* $\varepsilon^{\otimes I}$ by the formula

$$\varepsilon^{\otimes I} := \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}$$

(see the proof of Proposition 8) and for an integer $m \in \mathbb{Z}_{\geq 1}$, denote by $[m]$ the set $\{1, \dots, m\}$. Then the set

$$\left\{\varepsilon^{\otimes I} \mid I \in [n]^{[k]}\right\}$$

is a basis for $T^k(V^*)$; in particular, we have

$$\dim_{\mathbb{R}} T^k(V^*) = n^k.$$

Therefore, every covariant k-tensor $\alpha \in T^k(V^*)$ can be written uniquely in the form

$$\alpha = \alpha_I \, \varepsilon^{\otimes I} = \alpha_{i_1 \dots i_k} \, \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k},$$

where the n^k coefficients $\alpha_I = \alpha_{i_1 \dots i_k}$ are determined by

$$\alpha_{i_1 \dots i_k} = \alpha(E_{i_1}, \dots, E_{i_k}).$$

For example, $T^2(V^*)$ is the space of *bilinear forms* on V – note that a covariant 2-tensor on V is simply a real-valued bilinear function of two vectors – and every bilinear form on V can be written as $\beta = \beta_{ij} \, \varepsilon^i \otimes \varepsilon^j$ for some uniquely determined $n \times n$ matrix (β_{ij}).

Definition 11. For a covariant k-tensor $\alpha \in T^k(V^*)$ and a permutation $\sigma \in S_k$, denote by ${}^\sigma\alpha$ the covariant k-tensor given by

$$\begin{aligned} {}^\sigma\alpha \colon V \times \cdots \times V &\to \mathbb{R} \\ (v_1, \dots, v_k) &\mapsto \alpha\big(v_{\sigma(1)}, \dots, v_{\sigma(k)}\big). \end{aligned}$$

In the following two sections we will discuss two important subspaces of $T^k(V^*)$, namely the subspaces of *symmetric* resp. *alternating* covariant k-tensors. Both are described by the way that a permutation of the arguments of the given covariant k-tensor changes its value. A significant application of symmetric tensors in the theory of smooth manifolds is in the form of *Riemannian metrics*. Loosely speaking, a Riemannian metric is a choice of an inner product on each tangent space of the given manifold, varying smoothly from point to point, and allows one to define geometric concepts such as lenghts, angles and distances on the manifold. Riemannian metrics will not be discussed in this course, and this is the main reason why the discussion about symmetric tensors in section 3 will be kept to a minimum. On the other hand, *differential forms* will be discussed thoroughly in *Lecture 13* and *Lecture 14* of this course. They constitute a significant application of alternating tensors in smooth manifold theory, and they will be presented in section 4.

3 Symmetric Tensors

In all probability, you have already encountered the concept of *inner product* on a finite-dimensional real vector space V. It is a bilinear map $\langle\cdot,\cdot\rangle\colon V\times V\to\mathbb{R}$ which is symmetric and positive definite; in particular, $\langle\cdot,\cdot\rangle$ is a covariant 2-tensor on V, having the additional property that its value is unchanged when the two input arguments are exchanged; namely, we have $\langle v_1,v_2\rangle=\langle v_2,v_1\rangle$ for any $v_1,v_2\in V$. We now generalize this notion to any covariant k-tensor on V.

Definition 12. Let V be a finite-dimensional real vector space.

(a) A covariant k-tensor $\alpha\in T^k(V^*)$ on V is said to be *symmetric* if its value is unchanged by interchanging any pair of its arguments; namely, for all $v_1,\dots,v_k\in V$ and all $1\le i<j\le k$, we have

$$\alpha(v_1,\dots,v_i,\dots,v_j,\dots,v_k)=\alpha(v_1,\dots,v_j,\dots,v_i,\dots,v_k).$$

(b) The set of symmetric covariant k-tensors on V is denoted by $\Sigma^k(V^*)$. It is clearly a linear subspace of $T^k(V^*)$. By convention, we define $\Sigma^0(V^*):=\mathbb{R}$, and we also note that $\Sigma^1(V^*)=T^1(V^*)=V^*$.

Exercise 13. We define a projection $\mathrm{Sym}\colon T^k(V^*)\to\Sigma^k(V^*)$, called *symmetrization*, by the formula

$$\mathrm{Sym}(\alpha):=\frac{1}{k!}\sum_{\sigma\in S_k}{}^{\sigma}\alpha,$$

where ${}^{\sigma}\alpha$ was defined in Definition 11. Show that Sym is well-defined and linear, and that the following are equivalent:

(a) α is symmetric,

(b) $\alpha={}^{\sigma}\alpha$ for all $\sigma\in S_k$,

(c) $\alpha=\mathrm{Sym}(\alpha)$.

4 Alternating Tensors

Recall that the determinant may be regarded as a function $\det\colon \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$, taking as input n column vectors with n entries each, and having as output the determinant of the $n \times n$ matrix formed by these n column vectors. This map is multilinear, so det is a covariant n-tensor on $\mathbb{R}^n$. Moreover, it has the property that its value changes sign whenever two of its input entries are interchanged; in other words, det is an *alternating* n-tensor. We now generalize this notion to arbitrary covariant k-tensors.

Definition 14. Let V be a finite-dimensional real vector space.

(a) A covariant k-tensor $\alpha \in T^k(V^*)$ on V is said to be *alternating* (or *anti-symmetric* or *skew-symmetric*) if its value changes sign whenever any two of its arguments are interchanged; namely, for all $v_1, \ldots, v_k \in V$ and $1 \leq i < j \leq k$, we have

$$\alpha(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\alpha(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

(b) The set of alternating covariant k-tensors on V is denoted by $\Lambda^k(V^*)$. It is clearly a linear subspace of $T^k(V^*)$ and its elements of $\Lambda^k(V^*)$ are also called *exterior forms*, *multicovectors* or *k-covectors*. By convention, we define $\Lambda^0(V^*) := \mathbb{R}$, and we also note that $\Lambda^1(V^*) = T^1(V^*) = V^*$.

Note that every covariant 2-tensor β can be expressed as a sum of an alternating and a symmetric tensor, because

$$\begin{aligned}\beta(v,w) &= \frac{1}{2}\big(\beta(v,w) - \beta(w,v)\big) + \frac{1}{2}\big(\beta(v,w) + \beta(w,v)\big)\\ &= \alpha(v,w) + \sigma(v,w),\end{aligned}$$

where

$$\alpha(v,w) := \frac{1}{2}\big(\beta(v,w) - \beta(w,v)\big) \in \Lambda^2(V^*)$$

is an alternating 2-tensor on V and

$$\sigma(v,w) := \frac{1}{2}\big(\beta(v,w) + \beta(w,v)\big) \in \Sigma^2(V^*)$$

is a symmetric 2-tensor on V. However, this is not true for tensors of higher rank, as the following exercise demonstrates.

Exercise 15. Let (e^1, e^2, e^3) be the standard dual basis for $(\mathbb{R}^3)^*$. Show that $e^1 \otimes e^2 \otimes e^3$ is not equal to a sum of an alternating tensor and a symmetric tensor.

Recall that there is a group homomorphism $\operatorname{sgn}\colon S_k \to \{\pm 1\}$, which maps a permutation $\sigma \in S_k$ to 1 if it is a product of an even number of transpositions (even permutation), and to -1 otherwise (odd permutation). We may use it to describe alternating tensors as follows.

Exercise 16. We define a projection $\operatorname{Alt}\colon T^k(V^*) \to \Lambda^k(V^*)$, called *alternation*, by the formula

$$\operatorname{Alt}(\alpha) := \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, {}^{\sigma}\alpha,$$

where ${}^{\sigma}\alpha$ was defined in Definition 11. Show that Alt is well-defined and linear, and that the following are equivalent:

(a) α is alternating,

(b) $\alpha = (\operatorname{sgn}\sigma)\,{}^{\sigma}\alpha$ for all $\sigma \in S_k$,

(c) $\alpha = \operatorname{Alt}(\alpha)$,

(d) $\alpha(v_1, \dots, v_k) = 0$ whenever $v_1, \dots, v_k \in V$ are linearly dependent,

(e) $\alpha(v_1, \dots, v_k) = 0$ whenever there are $i \neq j$ such that $v_i = v_j$.

Example 17. Let us explicitly compute Alt for 1-, 2- and 3-tensors.

- If α is a 1-tensor, then $\operatorname{Alt}(\alpha) = \alpha$.
- If β is a 2-tensor, then
$$\operatorname{Alt}(\beta)(u, v) = \frac{1}{2}\big(\beta(u, v) - \beta(v, u)\big).$$
- If γ is a 3-tensor, then
$$\begin{aligned}\operatorname{Alt}(\gamma)(u, v, w) = \frac{1}{6}\big(&\gamma(u, v, w) + \gamma(v, w, u) + \gamma(w, u, v)\\ &-\gamma(v, u, w) - \gamma(u, w, v) - \gamma(w, v, u)\big).\end{aligned}$$

4.1 Elementary Alternating Tensors

Recall that for any basis of V, we described an induced basis of $T^k(V^*)$ in terms of tensor products of elements of the dual basis; cf. Proposition 8. We obtain here a similar description for a basis of $\Lambda^k(V^*)$.

Let V be a real vector space of dimension n, let $(E_1, \dots, E_n)$ be a basis for V, and denote by $(\varepsilon^1, \dots, \varepsilon^n)$ the corresponding dual basis for V^*. For a multi-index $I = (i_1, \dots, i_k) \in [n]^{[k]}$, define the *elementary alternating k-tensor* (or *elementary k-covector*) ε^I by the formula
$$\varepsilon^I := k! \operatorname{Alt}\left(\varepsilon^{\otimes I}\right),$$
where
$$\varepsilon^{\otimes I} = \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} \in T^k(V^*)$$
is the elementary k-tensor. Therefore, if $v_1, \dots, v_k \in V$, then the value of ε^I at the k-tuple $(v_1, \dots, v_k)$ is given by the formula
$$\begin{aligned}\varepsilon^I(v_1, \dots, v_k) &= \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, \varepsilon^{\otimes I}\left(v_{\sigma(1)}, \dots, v_{\sigma(k)}\right)\\ &= \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma) \prod_{1 \le j \le k} \varepsilon^{i_j}\left(v_{\sigma(j)}\right)\\ &= \det\begin{pmatrix} \varepsilon^{i_1}(v_1) & \cdots & \varepsilon^{i_1}(v_k)\\ \vdots & \ddots & \vdots\\ \varepsilon^{i_k}(v_1) & \cdots & \varepsilon^{i_k}(v_k)\end{pmatrix}.\end{aligned}$$

In other words, to compute $\varepsilon^I(v_1, \dots, v_k)$, we write the coefficients of $(v_1, \dots, v_k)$ with respect to the basis $(E_1, \dots, E_n)$ of V in the form of a $n \times k$-matrix, we consider the $k \times k$ submatrix formed by the lines $i_1, \dots, i_k$, and then we compute its determinant.

Example 18. In terms of the standard dual basis (e^1, e^2, e^3) for $(\mathbb{R}^3)^*$, we have

$$e^{13}(v,w) = \det \begin{pmatrix} v^1 & w^1 \\ v^3 & w^3 \end{pmatrix} = v^1 w^3 - v^3 w^1,$$

since $v = v^1 e_1 + v^2 e_2 + v^3 e_3$ and $w = w^1 e_1 + w^2 e_2 + w^3 e_3$, and

$$e^{123}(v,w,z) = \det(v,w,z).$$

Since $\operatorname{Alt}\colon T^k(V^*) \to \Lambda^k(V^*)$ is surjective, we know that $\{\varepsilon^I \mid I \in [n]^{[k]}\}$ is a generating set of $\Lambda^k(V^*)$. To extract from it a basis of $\Lambda^k(V^*)$, we need the following lemma, which describes the redundancy of $\{\varepsilon^I \mid I \in [n]^{[k]}\}$. In order to state it nicely, we need to introduce the following notation: for a multi-index $I \in [n]^{[k]}$ and a permutation $\sigma \in S_k$, denote by I_σ the multi-index

$$I_\sigma = \big(i_{\sigma(1)}, \dots, i_{\sigma(k)}\big).$$

Also, denote by δ^I_J the following generalization of the Kronecker-delta to multi-indices $I, J \in [n]^{[k]}$:

$$\delta^I_J := \begin{cases} \operatorname{sgn}\sigma & \text{if neither } I \text{ nor } J \text{ have repeated entries and } J = I_\sigma \text{ for some } \sigma \in S_k, \\ 0 & \text{if } I \text{ or } J \text{ have repeated entries or } J \text{ is not a permutation of } I. \end{cases}$$

and observe that

$$\delta^I_J = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \dots & \delta^{i_1}_{j_k} \\ \vdots & \ddots & \vdots \\ \delta^{i_k}_{j_1} & \dots & \delta^{i_k}_{j_k} \end{pmatrix}.$$

Lemma 19. *With the same notation as in the preceeding paragraph, the following statements hold:*

(a) *If I has a repeated index, then $\varepsilon^I = 0$.*

(b) *If $J = I_\sigma$ for some $\sigma \in S_k$, then $\varepsilon^J = (\operatorname{sgn}\sigma)\,\varepsilon^I$.*

(c) *For $I, J \in [n]^{[k]}$ we have*

$$\varepsilon^I\big(E_{j_1}, \dots, E_{j_k}\big) = \delta^I_J.$$

Proof. Exercise! □

Lemma 19 tells us that from the generating set $\{\varepsilon^I \mid I \in [n]^{[k]}\}$ of $\Lambda^k(V^*)$, we may discard all those ε^I's for which I has a repeated index, and for any I having no repeated index, we need only take one element from the set $\{\varepsilon^{I_\sigma} \mid \sigma \in S_k\}$ and discard the rest. A nice choice is thus the following: notice that for any multi-index I having no repeated indices, there exists a unique permutation $\sigma \in S_k$ such that I_σ is *strictly increasing*, i.e., $i_{\sigma(1)} < \dots < i_{\sigma(k)}$. Therefore, according to Lemma 19, the set $\{\varepsilon^I \mid I \in [n]^{[k]} \text{ is strictly increasing}\}$ still generates $\Lambda^k(V^*)$, and there is no obvious redundancy in it. Essentially due to Lemma 19(c), this set is linearly independent, and thus we obtain the following result:

Proposition 20. *With the same notation as above, the set*

$$\{\varepsilon^I \mid I \in [n]^{[k]} \text{ is a strictly increasing multi-index}\}$$

is a basis for $\Lambda^k(V^*)$. *In particular, we have*

$$\dim_{\mathbb{R}} \Lambda^k(V^*) = \binom{n}{k} = \frac{n!}{k!(n-k)!},$$

and

$$\Lambda^k(V^*) = \{0\} \quad \text{for } k > n.$$

Proof. Assume first that $k > n$. Since then every k-tuple of vectors is linearly dependent, it follows from Exercise 16(d) that $\Lambda^k(V^*) = \{0\}$.

Assume now that $k \leq n$. We need to show that

$$\mathcal{E} \coloneqq \{\varepsilon^I \mid I \in [n]^{[k]} \text{ is a strictly increasing multi-index}\}$$

is linearly independent and spans $\Lambda^k(V^*)$. The fact that $\mathcal{E}$ generates $\Lambda^k(V^*)$ was already discussed above. Suppose now that we have some linear relation

$$\sum_{I \in [n]^{[k]} \text{ strictly increasing}} \lambda_I \, \varepsilon^I = 0$$

for some $\lambda_I \in \mathbb{R}$. If we fix a strictly increasing multi-index $J \in [n]^{[k]}$, then evaluating the above relation at $(E_{j_1}, \ldots, E_{j_k})$ gives $\lambda_J = 0$ according to Lemma 19(c). Thus, $\mathcal{E}$ is linearly independent. In conclusion, $\mathcal{E}$ is a basis of $\Lambda^k(V^*)$, as desired. □

In particular, if V is a real vector space of dimension n, then the above proposition implies that $\Lambda^n(V^*)$ is 1-dimensional, spanned by the elementary n-covector $\varepsilon^{(1,\ldots,n)}$. As discussed in the beginning of this subsection, $\varepsilon^{(1,\ldots,n)}$ sends an n-tuple $(v_1, \ldots, v_n)$ to the determinant of the matrix $(v_j^i)_{1 \leq i,j, \leq n}$, where $v_j^i = \varepsilon^i(v_j)$ is the i-th component of v_j with respect to the chosen basis of V. Note that when $V = \mathbb{R}^n$ with the standard basis, the covector $\varepsilon^{(1,\ldots,n)}$ (which by definition is a function from $(\mathbb{R}^n)^n = \mathbb{R}^{n^2}$ to $\mathbb{R}$) is precisely the usual determinant function.

One consequence of this observation is the following useful description of the behavior of an n-covector on an n-dimensional vector space under linear maps. Recall that if $T \colon V \to V$ is a linear map, then the *determinant* of T is defined to be the determinant of the matrix representation of T with respect to any basis (recall that any two such matrix representation are conjugations of each other and hence have the same determinant, so this is well-defined).

Proposition 21. *Let V be an n-dimensional real vector space and let $\omega \in \Lambda^n(V^*)$. If $T \colon V \to V$ is any linear map and if $v_1, \ldots, v_n \in V$ are arbitrary vectors, then*

$$\omega(Tv_1, \ldots, Tv_n) = (\det T)\, \omega(v_1, \ldots, v_n). \tag{$\bullet$}$$

Proof. Let (E_i) be any basis for V, and let (ε^i) be the dual basis. Denote by $(T_i^j)_{1 \leq i,j \leq n}$ the matrix of T with respect to this basis, and set $T_i = TEi = \sum_j T_i^j E_j$. By Proposition 20, we can write $\omega = c\varepsilon^{(1,\ldots,n)}$ for some $c \in \mathbb{R}$. Since both sides of $(\bullet)$ are multilinear functions of $(v_1, \dots, v_n)$, it suffices to verify the identity when the v_i's are basis vectors. Furthermore, since both sides are alternating, by Lemma 19 we only need to check the case $(v_1, \dots, v_n) = (E_1, \dots, E_n)$. In this case, the right-hand side of $(\bullet)$ is

$$(\det T)\, c\, \varepsilon^{(1,\dots,n)}(E_1, \dots, E_n) = c \det T.$$

On the other hand, the left-hand side of $(\bullet)$ reduces to

$$\omega(TE_1, \dots, TE_n) = c\, \varepsilon^{(1,\dots,n)}(T_1, \dots, T_n) = c \det\big((\varepsilon^j(T_i))_{1\le i,j\le n}\big) = c \det\big((T_i^j)_{1\le i,j\le n}\big).$$

which is thus equal to the right-hand side. □

4.2 The Wedge Product

Recall that for any covariant tensors $\alpha \in T^k(V^*)$ and $\beta \in T^l(V^*)$ we defined the covariant $(k+l)$-tensor $\alpha \otimes \beta$; see Definition 6. This allowed us to build 'higher' covariant tensors out of lower ones, and also to describe a basis for $T^k(V^*)$ in terms of tensor products of elements of a dual basis. We now describe a similar construction for alternating tensors.

Definition 22. Let V be a finite-dimensional real vector space, and let $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$ be alternating tensors on V. The *wedge product* (or *exterior product*) of ω and η is denoted by $\omega \wedge \eta$ and is defined to be the $(k+l)$-covector given by the formula

$$\omega \wedge \eta \coloneqq \frac{(k+l)!}{k!l!}\operatorname{Alt}(\omega \otimes \eta).$$

As $\otimes$ is bilinear and Alt is linear, the map $-\wedge -\colon \Lambda^k(V^*) \times \Lambda^l(V^*) \to \Lambda^{k+l}(V^*)$ is bilinear. It is therefore natural to examine what the wedge product looks like on basis vectors. This also motivates the somewhat mysterious normalization factor $(k+l)!/(k!l!)$, because we have the following result.

Lemma 23. *Let V be a finite-dimensional real vector space, and let $(\varepsilon^1, \dots, \varepsilon^n)$ be a basis for V^*. For any multi-indices $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_l)$ we have the formula*

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{I \frown J},$$

where $I \frown J = (i_1, \dots, i_k, j_1, \dots, j_l)$ is the $(k+l)$-multi-index obtained by concatenating I and J.

Proof. By multilinearity, as in the proof of Proposition 8, it suffices to show that

$$\varepsilon^I \wedge \varepsilon^J(E_{p_1}, \dots, E_{p_{k+l}}) = \varepsilon^{I\frown J}(E_{p_1}, \dots, E_{p_{k+l}}) \tag{$\star$}$$

for any sequence of basis vectors $(E_{p_1}, \dots, E_{p_{k+l}})$. We do this by considering several cases.

Case 1: *The multi-index $P = (p_1, \dots, p_{k+l})$ has a repeated index.* Then by part (e) of Exercise 16, both sides of $(\star)$ evaluate to 0.

Case 2: *P contains an index that does not appear in either I or J.* In this case, the right-hand side of $(\star)$ is zero by part (c) of Lemma 19. Similarly, each term in the expansion of the left-hand side of $(\star)$ involves either I or J evaluated on a sequence of basis vectors that is not a permutation of I or J, respectively, so the left-hand side is also zero.

Case 3: *$P = I \frown J$ and P has no repeated indices.* In this case, the right-hand side of ($\star$) is equal to 1, again by part (c) of Lemma 19, so we need to show that the left-hand side is also equal to 1. By definition,

$$\begin{aligned}
\varepsilon^I \wedge \varepsilon^J(E_{p_1}, \dots, E_{p_{k+l}}) &= \\
&= \frac{(k+l)!}{k!l!} \operatorname{Alt}(\varepsilon^I \otimes \varepsilon^J) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) \varepsilon^I(E_{p_{\sigma(1)}}, \dots, E_{p_{\sigma(k)}}) \varepsilon^J(E_{p_{\sigma(k+1)}}, \dots, E_{p_{\sigma(k+l)}}).
\end{aligned}$$

By Lemma 19 again, the only terms in the sum above that give nonzero values are those in which σ permutes the first k indices and the last l indices of P separately. In other words, σ must be of the form $\sigma = \tau\eta$, where $\tau \in S_k$ acts by permuting $\{1, \dots, k\}$ and $\eta \in S_l$ acts by permuting $\{k+1, \dots, k+l\}$. Since then $\operatorname{sgn} \sigma = (\operatorname{sgn} \tau)(\operatorname{sgn} \eta)$, we have

$$\begin{aligned}
&\varepsilon^I \wedge \varepsilon^J(E_{p_1}, \dots, E_{p_{k+l}}) = \\
&= \frac{1}{k!l!} \sum_{\substack{\tau \in S_k \\ \eta \in S_l}} (\operatorname{sgn} \tau)(\operatorname{sgn} \eta)\, \varepsilon^I(E_{p_{\tau(1)}}, \dots, E_{p_{\tau(k)}})\, \varepsilon^J(E_{p_{k+\eta(1)}}, \dots, E_{p_{k+\eta(l)}}) \\
&= \left(\frac{1}{k!} \sum_{\tau \in S_k} (\operatorname{sgn} \tau)\, \varepsilon^I(E_{p_{\tau(1)}}, \dots, E_{p_{\tau(k)}}) \right) \left(\frac{1}{l!} \sum_{\eta \in S_l} (\operatorname{sgn} \eta)\, \varepsilon^J(E_{p_{k+\eta(1)}}, \dots, E_{p_{k+\eta(l)}}) \right) \\
&= \big(\operatorname{Alt}(\varepsilon^I)(E_{p_1}, \dots, E_{p_k})\big) \big(\operatorname{Alt}(\varepsilon^J)(E_{p_{k+1}}, \dots, E_{p_{k+l}})\big) \\
&= \varepsilon^I(E_{p_1}, \dots, E_{p_k})\, \varepsilon^J(E_{p_{k+1}}, \dots, E_{p_{k+l}}) \\
&= 1
\end{aligned}$$

where we used that Alt fixes alternating tensors by Exercise 16, and again used part (c) of Lemma 19 (recall that we are in the case $P = I \frown J$).

Case 4: *P is a permutation of $I \frown J$ and has no repeated indices.* In this case, applying a permutation to P brings us back to Case 3. As both sides of ($\star$) are alternating, the effect of this permutation is to multiply both sides by the same sign. Hence the result holds in this final case as well.

This completes the proof of the lemma. □

Together with the bilinearity of $- \wedge -$, this gives the following properties of the wedge product.

Proposition 24. *Let ω, η, ξ be multicovectors on a finite-dimensional real vector space V. Then we have the following properties:*

(a) Associativity:

$$\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi.$$

(b) Anticommutativity: *if $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$, then*

$$\omega \wedge \eta = (-1)^{kl}\, \eta \wedge \omega.$$

(c) *If* $(\varepsilon^1, \ldots, \varepsilon^n)$ *is a basis of* V^* *and* $I = (i_1, \ldots, i_k)$ *a multi-index, then*

$$\varepsilon^{i_1} \wedge \ldots \wedge \varepsilon^{i_k} = \varepsilon^I.$$

(d) *For any* $\omega^1, \ldots, \omega^k \in V^*$ *and* $v_1, \ldots, v_k \in V$ *we have*

$$\omega^1 \wedge \ldots \wedge \omega^k(v_1, \ldots, v_k) = \det\left(\left(\omega^j(v_i)\right)_{1 \leq i,j \leq k}\right).$$

Proof. Exercise! □

Due to Proposition 24(c), we generally use the notations ε^I and $\varepsilon^{i_1} \wedge \ldots \wedge \varepsilon^{i_k}$ interchangably.

An element $\eta \in \Lambda^k(V^*)$ is said to be *decomposable* if it can be expressed in the form $\eta = \omega^1 \wedge \ldots \wedge \omega^k$ for some covectors $\omega^1, \ldots, \omega^k \in V^*$. Note that not every k-covector is decomposable when $k > 1$; however, it follows from Proposition 20 and Proposition 24(c) that every k-covector can be written as a linear combination of decomposable ones.