EPFL

Differential Geometry II - Smooth Manifolds
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Exercise Sheet 14

Definition. Let M be a smooth manifold with or without boundary.

(a) A *curve segment in M* is defined to be a continuous curve $\gamma\colon [a,b] \to M$ whose domain is a compact interval. It is a *smooth curve segment in M* if it is smooth when $[a,b]$ is considered as a manifold with boundary (or, equivalently, if γ has an extension to a smooth curve defined in a neighborhood of each endpoint). It is a *piecewise smooth curve segment in M* if there exists a finite partition $a_0 = a < a_1 < \cdots < a_{k-1} < a_k = b$ of $[a,b]$ such that $\gamma|_{[a_{i-1},a_i]}$ is smooth[1] for every $1 \leq i \leq k$.

(b) Let ω be a smooth covector field on M. If $\gamma\colon [a,b] \to M$ is a piecewise smooth curve segment, then *the line integral of ω over γ* is defined to be the real number

$$\int_\gamma \omega := \sum_{i=1}^{k} \int_{[a_{i-1},a_i]} \gamma^*\omega,$$

where $[a_{i-1},a_i]$, $1 \leq i \leq k$, are subintervals of $[a,b]$ on which γ is smooth. If t denotes the standard coordinate on $\mathbb{R}$, then the smooth covector field $\omega_i := \gamma^*\omega = \big(\gamma|_{[a_{i-1},a_i]}\big)^*\omega$ on $[a_{i-1},a_i]$ can be written as $\omega_i = f_i(t)\,dt$ for some smooth function $f_i\colon [a_{i-1},a_i] \to \mathbb{R}$, so the integral of ω_i over $[a_{i-1},a_i]$ is given by

$$\int_{[a_{i-1},a_i]} \omega_i = \int_{a_{i-1}}^{a_i} f_i(t)\,dt.$$

Therefore,

$$\int_\gamma \omega = \sum_{i=1}^{k} \int_{a_{i-1}}^{a_i} f_i(t)\,dt.$$

[1]Continuity of γ means that $\gamma(t)$ approaches the same value as t approaches any of the points a_i (other than a_0 or a_k) from the left or the right. Smoothness of γ in each subinterval means that γ has one-sided velocity vectors at each such a_i when approaching from the left or the right, but these one-sided velocities need not be equal.

Exercise 1 (*Properties of line integrals*):
Let M be a smooth manifold with or without boundary. Let $\gamma\colon [a,b] \to M$ be a piecewise smooth curve segment in M, and let $\omega, \omega_1, \omega_2 \in \mathfrak{X}^*(M)$. Prove the following assertions:

(a) For any $c_1, c_2 \in \mathbb{R}$ we have
$$\int_\gamma \big(c_1\omega_1 + c_2\omega_2\big) = c_1 \int_\gamma \omega_1 + c_2 \int_\gamma \omega_2.$$

(b) If γ is a constant map, then
$$\int_\gamma \omega = 0.$$

(c) If $\gamma_1 := \gamma|_{[a,c]}$ and $\gamma_2 := \gamma|_{[c,b]}$, where $a, b, c \in \mathbb{R}$ with $a < c < b$, then
$$\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega.$$

(d) If $F\colon M \to N$ is any smooth map and if $\eta \in \mathfrak{X}^*(N)$, then
$$\int_\gamma F^*\eta = \int_{F\circ\gamma} \eta.$$

Exercise 2:
Let $M = \mathbb{R}^2\backslash\{0\}$. Consider the smooth covector field ω on M given by
$$\omega = \frac{xdy - ydx}{x^2 + y^2}.$$

(a) Show that ω is closed.

(b) Compute the integral of ω over the smooth curve segment
$$\gamma\colon [0, 2\pi] \to M,\ t \mapsto (\cos t, \sin t).$$

(c) Deduce that ω is not exact.

(d) Let (r, θ) be polar coordinates on the right half-plane $H = \big\{(x,y) \mid x > 0\big\}$. Compute the polar coordinate expression for $\omega \in \mathfrak{X}^*(M)$.

Exercise 3:
Consider the covector field $\omega \in \mathfrak{X}^*(\mathbb{R}^3)$ given by
$$\omega = e^{y^2}\, dx + 2xye^{y^2}\, dy - 2z\, dz.$$

(a) Show that ω is closed.

(b) Using the fact (*Poincaré's lemma*) that ω is exact on (the star-shaped set) $\mathbb{R}^3$, find a *potential* for ω, i.e., a function $f \in C^\infty(\mathbb{R}^3)$ such that $\omega = df$.

Definition. Let M be a smooth manifold with or without boundary. If $\gamma\colon [a,b] \to M$ and $\widetilde{\gamma}\colon [c,d] \to M$ are piecewise smooth curve segments in M, then we say that $\widetilde{\gamma}$ is a *reparametrization* of γ if $\widetilde{\gamma} = \gamma \circ \varphi$ for some diffeomorphism $\varphi\colon [c,d] \to [a,b]$. If φ is an increasing function, then we say that $\widetilde{\gamma}$ is a *forward reparametrization*, while if φ is a decreasing function, then we say that $\widetilde{\gamma}$ is a *backward reparametrization.* (More generally, with obvious modifications one can allow φ to be piecewise smooth.)

Exercise 4 (*Parameter independence of line integrals*)**:** Let M be a smooth manifold with or without boundary, $\omega \in \mathfrak{X}^*(M)$, and let γ be a piecewise smooth curve segment in M. Show that for any reparametrization $\widetilde{\gamma}$ of γ we have

$$\int_{\widetilde{\gamma}} \omega = \begin{cases} \displaystyle\int_{\gamma} \omega & \text{if } \widetilde{\gamma} \text{ is a forward reparametrization,} \\ \\ \displaystyle -\int_{\gamma} \omega & \text{if } \widetilde{\gamma} \text{ is a backward reparametrization.} \end{cases}$$

Exercise 5:
Let M be a smooth manifold with or without boundary and let $\omega \in \mathfrak{X}^*(M)$. If $\gamma\colon [a,b] \to M$ is a piecewise smooth curve segment in M, then the line integral of ω over γ can also be expressed as the ordinary integral

$$\int_{\gamma} \omega = \int_a^b \omega_{\gamma(t)}\big(\gamma'(t)\big)\,dt.$$

[Hint: Assume first that γ is smooth and that its image $\gamma([a,b])$ is contained in the domain of a single smooth chart for M, and then apply formula $(\star_6)$ from the proof of *Proposition 8.12*. Finally, use the compactness of $[a,b]$ to deal with the general case.]

Definition. Let M be a smooth manifold. The *length* of a smooth curve segment $\gamma\colon [a,b] \to \mathbb{R}^n$, where $n > 0$, is defined to be the value of the (ordinary) integral

$$L(\gamma) := \int_a^b \big|\gamma'(t)\big|\,dt.$$

Exercise 6:
Show that there is no smooth covector field ω on $\mathbb{R}^n$ with the property that

$$\int_{\gamma} \omega = L(\gamma)$$

for every smooth curve segment γ in $\mathbb{R}^n$.